

Nested Inequalities Among Divergence Measures

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Abstract

In this paper we have considered an inequality having 11 divergence measures. Out of them three are logarithmic such as Jeffries-Kullback-Leiber [4] [5] *J-divergence*, Burbea-Rao [1] *Jensen-Shannon divergence* and Taneja [7] *arithmetic-geometric mean divergence*. The other three are non-logarithmic such as *Hellinger discrimination*, *symmetric χ^2 -divergence*, and *triangular discrimination*. Three more are considered are due to mean divergences. Pranesh and Johnson [6] and Jain and Srivastava [3] studied different kind of divergence measures. We have considered measures arising due to differences of single inequality having 11 divergence measures in terms of a sequence. Based on these differences we have obtained many inequalities. These inequalities are kept as nested or sequential forms. Some reverse inequalities and equivalent versions are also studied.

Key words: *J-divergence; Jensen-Shannon divergence; Arithmetic-Geometric divergence; Mean divergence measures; Information inequalities.*

AMS Classification: 94A17; 26A48; 26D07.

1 Introduction

Let

$$\Gamma_n = \left\{ P = (p_1, p_2, \dots, p_n) \mid p_i > 0, \sum_{i=1}^n p_i = 1 \right\}, n \geq 2,$$

be the set of all complete finite discrete probability distributions. Let us consider the two groups of divergence measures:

• Logarithmic divergence measures

$$I(P||Q) = \frac{1}{2} \left[\sum_{i=1}^n p_i \ln \left(\frac{2p_i}{p_i + q_i} \right) + \sum_{i=1}^n q_i \ln \left(\frac{2q_i}{p_i + q_i} \right) \right],$$

$$J(P||Q) = \sum_{i=1}^n (p_i - q_i) \ln \left(\frac{p_i}{q_i} \right)$$

and

$$T(P||Q) = \sum_{i=1}^n \left(\frac{p_i + q_i}{2} \right) \ln \left(\frac{p_i + q_i}{2\sqrt{p_i q_i}} \right).$$

• Non-logarithmic divergence measures

$$\Delta(P||Q) = \sum_{i=1}^n \frac{(p_i - q_i)^2}{p_i + q_i},$$

$$h(P||Q) = \frac{1}{2} \sum_{i=1}^n (\sqrt{p_i} - \sqrt{q_i})^2$$

and

$$\Psi(P||Q) = \sum_{i=1}^n \frac{(p_i - q_i)^2 (p_i + q_i)}{p_i q_i}$$

The logarithmic measures $I(P||Q)$, $J(P||Q)$ and $T(P||Q)$ are three classical divergence measures known in the literature on information theory and statistics are *Jensen-Shannon divergence*, *J-divergence* and *Arithmetic-Geometric mean divergence* respectively [7] [8]. The non-logarithmic measures $\Delta(P||Q)$, $h(P||Q)$ and $\Psi(P||Q)$ are respectively known as *triangular discrimination*, *Hellinger's divergence* and *symmetric chi-square divergence*. In 2005, the author [9] proved the following inequality among these six symmetric divergence measures:

$$\frac{1}{4} \Delta \leq I \leq h \leq \frac{1}{8} J \leq T \leq \frac{1}{16} \Psi. \quad (1)$$

The above inequality (1) admits many nonnegative differences among the divergence measures. Based on these non-negative differences, the author [9] proved the following result:

$$\begin{aligned} D_{I\Delta} &\leq \frac{2}{3} D_{h\Delta} \leq \left\{ \frac{2D_{hI}}{\frac{1}{2} D_{J\Delta}} \leq \frac{1}{3} D_{T\Delta} \right\} \leq D_{TJ} \leq \\ &\leq \frac{2}{3} D_{Th} \leq 2D_{Jh} \leq \frac{1}{6} D_{\Psi\Delta} \leq \\ &\leq \frac{1}{5} D_{\Psi I} \leq \frac{2}{9} D_{\Psi h} \leq \frac{1}{4} D_{\Psi J} \leq \frac{1}{3} D_{\Psi T}, \end{aligned} \quad (2)$$

where, for example $D_{I\Delta} := I - \frac{1}{4}\Delta$, $D_{TJ} := T - \frac{1}{8}J$, $D_{F\Psi} := \frac{1}{16}F - \frac{1}{8}\Psi$, etc. The proof of the inequalities (2) is based on the following two lemmas:

Lemma 1.1. *If the function $f : [0, \infty) \rightarrow \mathbb{R}$ is convex and normalized, i.e., $f(1) = 0$, then the f -divergence, $C_f(P||Q)$ given by*

$$C_f(P||Q) = \sum_{i=1}^n q_i f\left(\frac{p_i}{q_i}\right), \quad (3)$$

is nonnegative and convex in the pair of probability distribution $(P, Q) \in \Gamma_n \times \Gamma_n$.

Lemma 1.2. *Let $f_1, f_2 : I \subset \mathbb{R}_+ \rightarrow \mathbb{R}$ two generating mappings are normalized, i.e., $f_1(1) = f_2(1) = 0$ and satisfy the assumptions:*

- (i) f_1 and f_2 are twice differentiable on (a, b) ;
- (ii) there exists the real constants α, β such that $\alpha < \beta$ and

$$\alpha \leq \frac{f_1''(x)}{f_2''(x)} \leq \beta, \quad f_2''(x) > 0, \quad \forall x \in (a, b),$$

then we have the inequalities:

$$\alpha C_{f_2}(P||Q) \leq C_{f_1}(P||Q) \leq \beta C_{f_2}(P||Q).$$

The Lemma 1.1 is due to Csiszár [2] and the Lemma 1.2 is due to author [8]. The aim of this paper is to consider more measures in (1) and improve the inequalities given in (2). These measures are based on the some well-known mean divergences.

1.1 Mean Divergence Measures

Author [10] studied the following inequality

$$G(P||Q) \leq N_1(P||Q) \leq N_2(P||Q) \leq A(P||Q), \quad (4)$$

where

$$\begin{aligned} G(P||Q) &= \sum_{i=1}^n \sqrt{p_i q_i}, \\ N_1(P||Q) &= \sum_{i=1}^n \left(\frac{\sqrt{p_i} + \sqrt{q_i}}{2} \right)^2, \\ N_2(P||Q) &= \sum_{i=1}^n \sqrt{\frac{p_i + q_i}{2}} \left(\frac{\sqrt{p_i} + \sqrt{q_i}}{2} \right) \end{aligned}$$

and

$$A(P||Q) = \sum_{i=1}^n \frac{p_i + q_i}{2} = 1.$$

The above inequality admits non-negative differences given by

$$\begin{aligned} M_1(P||Q) &= D_{N_2 N_1}(P||Q) = \\ &= \sum_{i=1}^n \left(\sqrt{\frac{p_i + q_i}{2}} \left(\frac{\sqrt{p_i} + \sqrt{q_i}}{2} \right) - \left(\frac{\sqrt{p_i} + \sqrt{q_i}}{2} \right)^2 \right), \\ M_2(P||Q) &= D_{N_2 G}(P||Q) = \\ &= \sum_{i=1}^n \left[\left(\frac{\sqrt{p_i} + \sqrt{q_i}}{2} \right) \sqrt{\frac{p_i + q_i}{2}} - \sqrt{p_i q_i} \right], \\ M_3(P||Q) &= D_{A N_2}(P||Q) = \\ &= \sum_{i=1}^n \left[\left(\frac{p_i + q_i}{2} \right) - \left(\frac{\sqrt{p_i} + \sqrt{q_i}}{2} \right) \left(\sqrt{\frac{p_i + q_i}{2}} \right) \right] \end{aligned}$$

and

$$h(P||Q) = 2D_{A N_1}(P||Q) = D_{AG}(P||Q) = D_{N_1 G}(P||Q).$$

1.2 New Measures

Jain and Srivastava [3] and Kumar and Johnson [6] respectively studied the measures

$$K_0(P||Q) = \sum_{i=1}^n \frac{(p_i - q_i)^2}{\sqrt{p_i q_i}}$$

and

$$F(P||Q) = \frac{1}{2} \sum_{i=1}^n \frac{(p_i^2 - q_i^2)^2}{\sqrt{(p_i q_i)^3}}.$$

In total we have 11 divergence measures. By the application of Lemmas 1 and 2 we can put them in a single inequality as

$$\begin{aligned} \frac{1}{4}\Delta \leq I \leq 4M_1 \leq \frac{4}{3}M_2 \leq h \leq 4M_3 \leq \\ \leq \frac{1}{8}J \leq T \leq \frac{1}{8}K_0 \leq \frac{1}{16}\Psi \leq \frac{1}{16}F. \end{aligned} \quad (5)$$

1.3 Pyramid

The 11 measures appearing in the inequalities (5) admits 55 non-negative differences. These 55 non-negative difference satisfies some obvious inequalities given below in the form of **pyramid** or **triangular**:

1. $D_{I\Delta}^1$;
2. $D_{M_1 I}^2 \leq D_{M_1 \Delta}^3$;
3. $D_{M_2 M_1}^4 \leq D_{M_2 I}^5 \leq D_{M_2 \Delta}^6$;
4. $D_{h M_2}^7 \leq D_{h M_1}^8 \leq D_{h I}^9 \leq D_{h \Delta}^{10}$;
5. $D_{M_3 h}^{11} \leq D_{M_3 M_2}^{12} \leq D_{M_3 M_1}^{13} \leq D_{M_3 I}^{14} \leq D_{M_3 \Delta}^{15}$;

6. $D_{JM_3}^{16} \leq D_{Jh}^{17} \leq D_{JM_2}^{18} \leq D_{JM_1}^{19} \leq D_{JI}^{20} \leq D_{J\Delta}^{21}$;
7. $D_{TJ}^{22} \leq D_{TM_3}^{23} \leq D_{Th}^{24} \leq D_{TM_2}^{25} \leq D_{TM_1}^{26} \leq D_{TI}^{27} \leq D_{T\Delta}^{28}$;
8. $D_{K_0T}^{29} \leq D_{K_0J}^{30} \leq D_{K_0M_3}^{31} \leq D_{K_0h}^{32} \leq D_{K_0M_2}^{33} \leq D_{K_0M_1}^{34} \leq D_{K_0I}^{35} \leq D_{K_0\Delta}^{36}$;
9. $D_{\Psi K_0}^{37} \leq D_{\Psi T}^{38} \leq D_{\Psi J}^{39} \leq D_{\Psi M_3}^{40} \leq D_{\Psi h}^{41} \leq D_{\Psi M_2}^{42} \leq D_{\Psi M_1}^{43} \leq D_{\Psi I}^{44} \leq D_{\Psi\Delta}^{45}$;
10. $D_{FK_0}^{46} \leq D_{FK_0J}^{47} \leq D_{FK_0T}^{48} \leq D_{FK_0h}^{49} \leq D_{FK_0M_3}^{50} \leq D_{FK_0M_2}^{51} \leq D_{FK_0M_1}^{52} \leq D_{FK_0I}^{53} \leq D_{FK_0\Delta}^{54}$;

The following equalities hold:

$$\begin{aligned} D_{hM_1}^8 &= 3D_{hM_2}^7 = \frac{3}{2}D_{M_2M_1}^4 = \\ &= D_{M_3h}^{11} = \frac{1}{2}D_{M_3M_1}^{13} = \frac{3}{4}D_{M_3M_2}^{12} \end{aligned}$$

and

$$D_{TJ}^{22} = \frac{1}{2}D_{TI}^{27} = D_{JI}^{20}.$$

In view of Lemmas 1 and 2, the measures appearing in the above pyramid are convex in a pair of probability distributions and can be written as

$$D_{AB} := \sum_{i=1}^n q_i f_{AB} \left(\frac{p_i}{q_i} \right), \quad (6)$$

where $f_{AB}(x) = f_A(x) - f_B(x)$, $A \geq B$ with the property that $f''_{AB}(x) \geq 0$, $\forall x > 0$.

In this paper our aim is to extend the results given by (2) by taking all possible nonnegative differences given in the above pyramid. These inequalities we have put in nested or sequential form.

2 Nested Inequalities

In this section, we shall try to put the measures appearing in above pyramid in terms of nested or sequential form. This we have done in a theorem below.

Theorem 2.1. *The following inequalities hold:*

$$\begin{aligned} D_{I\Delta}^1 &\leq \frac{8}{9}D_{M_1\Delta}^3 \leq \frac{8}{11}D_{M_2\Delta}^6 \leq \frac{2}{3}D_{h\Delta}^{10} \leq \frac{8}{15}D_{M_3\Delta}^{15} \leq \\ &\leq \left\{ \left\{ \frac{1}{2}D_{J\Delta}^{21} \right\} \leq \frac{1}{3}D_{T\Delta}^{28} \right\} \leq \frac{8}{3}D_{M_2I}^5 \leq \\ &\leq \left\{ \frac{8}{3}D_{hM_1}^8 \leq \frac{8}{7}D_{M_3I}^{14} \leq 2D_{hI}^9 \right\} \end{aligned}$$

$$\begin{aligned} &\leq \left\{ \frac{1}{3}D_{K_0\Delta}^{36} \right. \\ &\quad \left. \left\{ \left\{ D_{TJ}^{22} \right\} \leq \frac{8}{15}D_{TM_1}^{26} \right\} \leq \frac{8}{13}D_{TM_2}^{25} \leq \frac{2}{3}D_{Th}^{24} \right\} \leq \\ &\quad \left. 8D_{M_1I}^2 \right\} \\ &\leq \left\{ \leq \frac{2}{5}D_{JM_2}^{18} \leq 2D_{Jh}^{17} \right. \\ &\quad \left. \frac{8}{9}D_{TM_3}^{23} \right\} \leq \frac{1}{2}D_{K_0I}^{35} \leq \frac{8}{15}D_{K_0M_1}^{34} \leq \\ &\leq \frac{8}{13}D_{K_0M_2}^{33} \leq \frac{2}{3}D_{K_0h}^{32} \leq \left\{ \left\{ D_{K_0J}^{30} \right\} \leq \frac{1}{5}D_{\Psi I}^{44} \right\} \leq \\ &\quad \left. \frac{8}{9}D_{K_0M_3}^{31} \right\} \\ &\leq \frac{8}{39}D_{\Psi M_1}^{43} \leq \frac{8}{37}D_{\Psi M_2}^{42} \leq \frac{2}{9}D_{\Psi h}^{41} \leq \left\{ \frac{1}{4}D_{\Psi J}^{39} \right\} \leq \\ &\quad \left. \frac{8}{33}D_{\Psi M_3}^{40} \right\} \\ &\leq \frac{1}{3}D_{\Psi K_0}^{37} \leq \left\{ \frac{1}{3}D_{\Psi T}^{38} \right\} \leq \frac{1}{8}D_{FI}^{54} \leq \frac{8}{63}D_{FM_1}^{53} \leq \\ &\leq \frac{8}{61}D_{FM_2}^{52} \leq \frac{2}{15}D_{Fh}^{51} \leq \left\{ \frac{1}{7}D_{FJ}^{49} \right\} \leq \\ &\quad \left. \frac{8}{57}D_{FM_3}^{50} \right\} \leq \\ &\leq \frac{1}{6}D_{FK_0}^{47} \leq \frac{1}{6}D_{FT}^{48} \leq \frac{1}{3}D_{F\Psi}^{46} \quad (7) \\ &\leq \left\{ \frac{1}{4}D_{Jh}^{17} \right\} \\ &\quad \left\{ D_{M_1I}^2 \right\} \leq D_{JM_3}^{16} \leq \frac{1}{24}D_{F\Psi}^{46}. \quad (8) \\ &\quad \left\{ \frac{1}{9}D_{TM_3}^{23} \right\} \end{aligned}$$

Proof. In view of (6) we shall prove the theorem just writing the expressions for $f_{AB}(x)$. The rest part is understood obviously.

1. For $D_{I\Delta}^1(P||Q) \leq \frac{8}{9}D_{M_1\Delta}^3(P||Q)$: After simplification, we observe that equivalently, we have to show the following:

$$I \leq \frac{1}{36} [128M_1 + \Delta]. \quad (9)$$

Let us consider the function $g_{I-M_1-\Delta}(x) = \frac{f''_I(x)}{f''_{M_1-\Delta}(x)}$, then we have

$$\begin{aligned} g_{I-M_1-\Delta}(x) &= \\ &= \frac{(x+1)^2 \sqrt{x} \sqrt{2x+2}}{16 \left(\sqrt{2x+2} (2(x+1)^3 + x^{3/2}) - \right. \\ &\quad \left. -2(x^{3/2} + 1)(x+1)^2 \right)}. \quad (10) \end{aligned}$$

Here we have

$$f_{M_1 \Delta}(x) = 128f_{M_1}(x) + f_{\Delta}(x).$$

Calculating the first order derivative of the function $g_{I-M_1 \Delta}(x)$, we get

$$\begin{aligned} g'_{I-M_1 \Delta}(x) &= \\ &= - \frac{\left(\sqrt{2x+2} (x+1) (\sqrt{x}-1) \times \right. \\ &\quad \left. \times (x^2 + x^{3/2} + 3x + \sqrt{x} + 1) \right) \times k_1(x)}{16\sqrt{x} \left(\sqrt{2x+2} (2(x+1)^3 + x^{3/2}) - \right. \\ &\quad \left. - 2(x^{3/2} + 1)(x+1)^2 \right)}, \end{aligned}$$

where

$$k_1(x) = \sqrt{2x+2} (x^{3/2} + 1) - (x+1)^2.$$

This gives

$$g'_{I-M_1 \Delta}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases}. \quad (11)$$

Expression (11) is valid only when $k_1(x) > 0$, $\forall x > 0$, $x \neq 1$. Now, we shall show that $k_1(x) > 0$, $\forall x > 0$, $x \neq 1$. Let us consider

$$h_1(x) = \left[\sqrt{2x+2} (x^{3/2} + 1) \right]^2 - (x+1)^4.$$

After simplifications, we have

$$h_1(x) = (x+1) (\sqrt{x}-1)^2 (x^2 + 2x^{3/2} + 2\sqrt{x} + 1).$$

Since $h_1(x) > 0$, $\forall x > 0$, $x \neq 1$ proving that $k_1(x) > 0$, $\forall x > 0$, $x \neq 1$. Also we have

$$\begin{aligned} \beta &= \sup_{x \in (0, \infty)} g_{I-M_1 \Delta}(x) = \\ &= \lim_{x \rightarrow 1} g_{I-M_1 \Delta}(x) = \frac{1}{36}. \end{aligned} \quad (12)$$

By the application Lemma 1.2 over (11) and (12) we get (9), proving the required result.

Argument: Let a and b two positive numbers, i.e., $a > 0$ and $b > 0$. If $a^2 - b^2 > 0$, then we can conclude that $a > b$ because $a - b = (a^2 - b^2)/(a + b)$. We have used this argument to prove $k_1(x) > 0$, $\forall x > 0$, $x \neq 1$. We shall use frequently this argument to prove the other parts of the theorem.

Remark 2.1. From the above proof we observe that it is sufficient to write expressions similar to (10), (11) and (12). The rest part of the proof follows by the application of Lemma 1.2. In view of this we shall avoid details for the proof of other parts. From now onward, throughout it is understood that $x > 0$, $x \neq 1$.

2. **For $D_{M_1 \Delta}^3(\mathbf{P}||\mathbf{Q}) \leq \frac{9}{11} D_{M_2 \Delta}^6(\mathbf{P}||\mathbf{Q})$:** Let us consider $g_{M_1 \Delta-M_2 \Delta}(x) = f''_{M_1 \Delta}(x)/f''_{M_2 \Delta}(x)$, then we have

$$\begin{aligned} g_{M_1 \Delta-M_2 \Delta}(x) &= \\ &= \frac{3 \left(\sqrt{2x+2} ((x+1)^3 - 4x^{3/2}) - \right. \\ &\quad \left. - (x^{3/2} + 1)(x+1)^2 \right)}{\left(2\sqrt{2x+2} [(x+1)^3 - 6x^{3/2}] - \right. \\ &\quad \left. - (x^{3/2} + 1)(x+1)^2 \right)}. \end{aligned}$$

This gives

$$\beta = \lim_{x \rightarrow 1} g_{M_1 \Delta-M_2 \Delta}(x) = \frac{9}{11}.$$

Equivalently, we have to show that

$$\begin{aligned} \Omega_1 &= \frac{9}{11} D_{M_2 \Delta}^6 - D_{M_1 \Delta}^3 = \\ &= \frac{1}{22} (\Delta + 24M_2 - 88M_1) \geq 0. \end{aligned}$$

We can write $\Omega_1 := \sum_{i=1}^n q_i f_{\Omega_1}(q_i/p_i)$, where $f_{\Omega_1}(x) = \frac{k_1(x)}{22(x+1)}$, with

$$\begin{aligned} k_2(x) &= 20x^{3/2} + 20\sqrt{x} + 23x^2 + 42x + 23 \\ &\quad - 16\sqrt{2x+2} (\sqrt{x} + 1)(x+1). \end{aligned}$$

Let us consider

$$\begin{aligned} h_2(x) &= (20x^{3/2} + 20\sqrt{x} + 23x^2 + 42x + 23)^2 \\ &\quad - [16\sqrt{2x+2} (\sqrt{x} + 1)(x+1)]^2. \end{aligned}$$

After simplifications, we get

$$h_2(x) = (\sqrt{x}-1)^6 [16x + 16 + (\sqrt{x}-1)^2].$$

Since $h_2(x) > 0$, proving that $k_2(x) > 0$. This proves the required result.

3. **For $D_{M_2 \Delta}^6(\mathbf{P}||\mathbf{Q}) \leq \frac{11}{12} D_{h \Delta}^{10}(\mathbf{P}||\mathbf{Q})$:** Let us consider $g_{M_2 \Delta-h \Delta}(x) = f''_{M_2 \Delta}(x)/f''_{h \Delta}(x)$, then we have

$$\begin{aligned} g_{M_2 \Delta-h \Delta}(x) &= \\ &= \frac{\left(2\sqrt{2x+2} [(x+1)^3 - 6x^{3/2}] - \right. \\ &\quad \left. - (x^{3/2} + 1)(x+1)^2 \right)}{\left(3\sqrt{2x+2} (\sqrt{x}-1)^2 \times \right. \\ &\quad \left. \times (x^2 + 2x^{3/2} + 6x + 2\sqrt{x} + 1) \right)}. \end{aligned}$$

This gives

$$\beta = \lim_{x \rightarrow 1} g_{M_2 \Delta-h \Delta}(x) = \frac{11}{12}.$$

Equivalently, we have to show that

$$\begin{aligned} \Omega_2 &= \frac{11}{12} D_{h \Delta}^{10} - D_{M_2 \Delta}^6 = \\ &= \frac{1}{48} (44h + \Delta - 64M_2) \geq 0. \end{aligned}$$

We can write $\Omega_2 := \sum_{i=1}^n q_i f_{\Omega_2}(q_i/p_i)$, where $f_{\Omega_2}(x) = \frac{k_3(x)}{48(x+1)}$, with $k_3(x) = k_2(x) > 0$. This proves the required result.

4. For $D_{h\Delta}^{10}(\mathbf{P}||\mathbf{Q}) \leq \frac{4}{5}D_{M_3\Delta}^{15}(\mathbf{P}||\mathbf{Q})$: Let us consider $g_{h\Delta-M_3\Delta}(x) = f''_{h\Delta}(x)/f''_{M_3\Delta}(x)$, then we have

$$g_{h\Delta-M_3\Delta}(x) = \frac{\left(\begin{aligned} &(x^2 + 2x^{3/2} + 6x + 2\sqrt{x} + 1) \times \\ &\times (\sqrt{x} - 1)^2 \sqrt{2x+2} \end{aligned} \right)}{2 \left[(x^{3/2} + 1)(x+1)^2 - 4x^{3/2}\sqrt{2x+2} \right]}.$$

This gives

$$\beta = \lim_{x \rightarrow 1} g_{h\Delta-M_3\Delta}(x) = \frac{4}{5}.$$

Equivalently, we have to show that

$$\begin{aligned} \Omega_3 &= \frac{4}{5}D_{M_3\Delta}^{15} - D_{h\Delta}^{10} = \\ &= \frac{1}{20}(64M_3 + \Delta - 20h) \geq 0. \end{aligned}$$

We can write $\Omega_3 := \sum_{i=1}^n q_i f_{\Omega_3}(q_i/p_i)$, where $f_{\Omega_3}(x) = \frac{k_4(x)}{20(x+1)}$, with $k_4(x) = k_2(x) > 0$. This proves the required result.

5. For $D_{M_3\Delta}^{15}(\mathbf{P}||\mathbf{Q}) \leq 5D_{hM_1}^8(\mathbf{P}||\mathbf{Q})$: Let us consider $g_{M_3\Delta-hM_1}(x) = f''_{M_3\Delta}(x)/f''_{hM_1}(x)$, then we have

$$\begin{aligned} g_{M_3\Delta-hM_1}(x) &= \\ &= \frac{2 \left[(x^{3/2} + 1)(x+1)^2 - 4x^{3/2}\sqrt{2x+2} \right]}{(x+1)^2 [2x^{3/2} + 2 - (x+1)\sqrt{2x+2}]}. \end{aligned}$$

This gives

$$\beta = \lim_{x \rightarrow 1} g_{M_3\Delta-hM_1}(x) = 5.$$

Equivalently, we have to show that

$$\begin{aligned} \Omega_4 &= \frac{4}{5}D_{M_3\Delta}^{15} - D_{h\Delta}^{10} = \\ &= \frac{1}{4}(20h + \Delta - 80M_1 - 64M_3) \geq 0. \end{aligned}$$

We can write $\Omega_4 := \sum_{i=1}^n q_i f_{\Omega_4}(q_i/p_i)$, where $f_{\Omega_4}(x) = \frac{k_5(x)}{4(x+1)}$, with $k_5(x) = k_2(x) > 0$. This proves the required result.

6. For $D_{M_3\Delta}^{15}(\mathbf{P}||\mathbf{Q}) \leq \frac{15}{16}D_{J\Delta}^{21}(\mathbf{P}||\mathbf{Q})$: Let us consider $g_{M_3\Delta-J\Delta}(x) = f''_{M_3\Delta}(x)/f''_{J\Delta}(x)$, then we have

$$g_{M_3\Delta-J\Delta}(x) = \frac{4\sqrt{x} \left(\begin{aligned} &(x^{(3/2)} + 1)(x+1)^2 - \\ &- 4x^{3/2}\sqrt{2x+2} \end{aligned} \right)}{\sqrt{2x+2}(x^2 + 6x + 1)(x-1)^2}$$

and

$$\begin{aligned} g'_{M_3\Delta-J\Delta}(x) &= \\ &= -\frac{2(x+1) \times k_6(x)}{\sqrt{2x+2}\sqrt{x}(x^2 + 6x + 1)^2(x-1)^3}, \end{aligned}$$

where

$$\begin{aligned} k_6(x) &= -16x^{3/2}\sqrt{2x+2}(x+1)^2 + \\ &+ (\sqrt{x} + 1) \left(\begin{aligned} &(x^4 + 9x^2 + 1)(\sqrt{x} - 1)^2 + \\ &+ x^{9/2} + x^4 + 2x^{7/2} + 28x^3 + \\ &+ 28x^2 + 2x^{3/2} + x + \sqrt{x} \end{aligned} \right). \end{aligned}$$

This gives

$$g'_{M_3\Delta-J\Delta}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases},$$

provided $k_6(x) > 0$. Now we shall show that $k_6(x) > 0$. Let us consider

$$\begin{aligned} h_6(x) &= - \left[16x^{3/2}\sqrt{2x+2}(x+1)^2 \right]^2 + \\ &+ \left[(\sqrt{x} + 1) \left(\begin{aligned} &(x^4 + 9x^2 + 1)(\sqrt{x} - 1)^2 + \\ &+ x^{9/2} + x^4 + 2x^{7/2} + 28x^3 + \\ &+ 28x^2 + 2x^{3/2} + x + \sqrt{x} \end{aligned} \right) \right]^2. \end{aligned}$$

Simplifying the above expression, we get

$$\begin{aligned} h_6(x) &= (\sqrt{x} - 1)^4 \times \\ &\times \left(\begin{aligned} &1 + 4\sqrt{x} + 538x^{5/2} + 36x^{3/2} + \\ &+ 1460x^{7/2} + 2196x^{9/2} + 538x^{13/2} + \\ &+ 1537x^5 + 1537x^4 + 908x^3 + 12x^8 \\ &+ 166x^2 + 12x + 4x^{17/2} + 36x^{15/2} + \\ &+ 1460x^{11/2} + x^9 + 908x^6 + 166x^7 \end{aligned} \right). \end{aligned}$$

Since $h_6(x) > 0$, proving that $k_6(x) > 0$. Also we have

$$\beta = \sup_{x \in (0, \infty)} g_{M_3\Delta-J\Delta}(x) = \lim_{x \rightarrow 1} g_{M_3\Delta-J\Delta}(x) = \frac{15}{16}.$$

7. For $D_{J\Delta}^{21}(\mathbf{P}||\mathbf{Q}) \leq \frac{2}{3}D_{T\Delta}^{28}(\mathbf{P}||\mathbf{Q})$: It holds in view of (2).

8. For $D_{hM_1}^8(\mathbf{P}||\mathbf{Q}) \leq \frac{1}{8}D_{T\Delta}^{28}(\mathbf{P}||\mathbf{Q})$: Let us consider $g_{hM_1-T\Delta}(x) = f''_{hM_1}(x)/f''_{T\Delta}(x)$, then we have

$$\begin{aligned} g_{hM_1-T\Delta}(x) &= \\ &= \frac{\sqrt{x}(x+1)^2 \left[2(x^{3/2} + 1) - \sqrt{2x+2}(x+1) \right]}{\sqrt{2x+2}(x^2 + 4x + 1)(x-1)^2} \end{aligned}$$

and

$$g'_{hM_1-T\Delta}(x) = -\frac{(x+1)k_7(x)}{\left(\begin{aligned} &2(x^2 + 4x + 1)^2 \times \\ &\times (x-1)^3 \sqrt{x}\sqrt{2x+2} \end{aligned} \right)},$$

where

$$\begin{aligned} k_7(x) &= \\ &= 2(\sqrt{x} + 1) \left(\begin{aligned} &(x^4 + 5x^2 + 1)(\sqrt{x} - 1)^2 + \\ &+ 3x^4 + x^{9/2} + 20x^3 + \\ &+ 20x^2 + 3x + \sqrt{x} \end{aligned} \right) - \\ &- \sqrt{2x+2}(x^4 + 6x^3 + 34x^2 + 6x + 1)(x+1). \end{aligned}$$

This give

$$g'_{hM_1-T\Delta}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases},$$

provided $k_7(x) > 0$. Now we shall show that $k_7(x) > 0$. Let us consider

$$\begin{aligned} h_7(x) &= \\ &= \left[2(\sqrt{x}+1) \left(\frac{(x^4+5x^2+1)(\sqrt{x}-1)^2+}{+3x^4+x^{9/2}+20x^3+} \right) \right]^2 \\ &\quad - \left[\sqrt{2x+2}(x+1) \left(\frac{x^4+6x^3+}{+34x^2+6x+1} \right) \right]^2. \end{aligned}$$

After simplifications, we get

$$\begin{aligned} h_7(x) &= 2(\sqrt{x}-1)^4 \times \\ &\times \left(\begin{aligned} &x^9+4x^{17/2}+7x^8+24x^{15/2}+44x^7+ \\ &+164x^{13/2}+104x^6+222x^{11/2}+ \\ &+42x^{7/2}(x+1)(\sqrt{x}-1)^2+300x^{9/2}+ \\ &+222x^{7/2}+104x^3+164x^{5/2}+ \\ &+44x^2+24x^{3/2}+7x+4\sqrt{x}+1 \end{aligned} \right). \end{aligned}$$

Since $h_7(x) > 0$, this gives that $k_7(x) > 0$. Also we have

$$\beta = \sup_{x \in (0, \infty)} g_{hM_1-T\Delta}(x) = \lim_{x \rightarrow 1} g_{hM_1-T\Delta}(x) = \frac{1}{8}.$$

9. For $\mathbf{D}_{M_3h}^{11}(\mathbf{P}||\mathbf{Q}) \leq \frac{3}{7}\mathbf{D}_{M_3I}^{14}(\mathbf{P}||\mathbf{Q})$: Let us consider $g_{M_3h-M_3I}(x) = f''_{M_3h}(x)/f''_{M_3I}(x)$, then we have

$$g_{M_3h-M_3I}(x) = \frac{2(x^{3/2}+1) - (x+1)\sqrt{2x+2}}{2[(x^{3/2}+1) - \sqrt{x}\sqrt{2x+2}]}$$

and

$$\begin{aligned} g'_{M_3h-M_3I}(x) &= \\ &= -\frac{(\sqrt{x}-1) \times k_8(x)}{2\sqrt{x}\sqrt{2x+2}[(x^{3/2}+1) - \sqrt{x}\sqrt{2x+2}]^2}, \end{aligned}$$

where

$$\begin{aligned} k_8(x) &= -(x+1)(\sqrt{x}+1)\sqrt{2x+2}+ \\ &+ (x+1)(\sqrt{x}-1)^2 + (x^{3/2}+1)(\sqrt{x}+1) + 4x. \end{aligned}$$

This gives

$$g'_{M_3h-M_3I}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases},$$

provided $k_8(x) > 0$. In order to prove $k_8(x) > 0$, let us consider

$$\begin{aligned} h_8(x) &= \left(\begin{aligned} &(x+1)(\sqrt{x}-1)^2+ \\ &+ (x^{3/2}+1)(\sqrt{x}+1) + 4x \end{aligned} \right)^2 - \\ &- [(x+1)(\sqrt{x}+1)\sqrt{2x+2}]^2. \end{aligned}$$

After simplifications, we have

$$h_8(x) = (2x+1)(x+2)(\sqrt{x}-1)^4.$$

Since $h_8(x) > 0$, this gives that $k_8(x) > 0$. Also we have

$$\beta = \sup_{x \in (0, \infty)} g_{M_3h-M_3I}(x) = \lim_{x \rightarrow 1} g_{M_3h-M_3I}(x) = \frac{3}{7}.$$

10. For $\mathbf{D}_{M_3I}^{14}(\mathbf{P}||\mathbf{Q}) \leq \frac{7}{4}\mathbf{D}_{M_3I}^9(\mathbf{P}||\mathbf{Q})$: Let us consider $g_{M_3I-hI}(x) = f''_{M_3I}(x)/f''_{M_3h}(x)$, then we have

$$g_{M_3I-hI}(x) = \frac{2[x^{3/2}+1 - \sqrt{x}\sqrt{2x+2}]}{(\sqrt{x}-1)^2\sqrt{2x+2}}$$

and

$$g'_{M_3I-hI}(x) = -\frac{k_9(x)}{(\sqrt{x}-1)^3(x+1)\sqrt{x}\sqrt{2x+2}},$$

where $k_9(x) = k_8(x) > 0$. This give

$$g'_{M_3I-hI}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases}.$$

Also we have

$$\beta = \sup_{x \in (0, \infty)} g_{M_3I-hI}(x) = \lim_{x \rightarrow 1} g_{M_3I-hI}(x) = \frac{7}{4}.$$

11. For $\mathbf{D}_{hI}^9(\mathbf{P}||\mathbf{Q}) \leq \frac{4}{3}\mathbf{D}_{M_2I}^5(\mathbf{P}||\mathbf{Q})$: Let us consider $g_{hI-M_2I}(x) = f''_{hI}(x)/f''_{M_2I}(x)$, then we have

$$\begin{aligned} g_{hI-M_2I}(x) &= \\ &= \frac{3(\sqrt{x}-1)^2\sqrt{2x+2}}{2[\sqrt{2x+2}(2x-3\sqrt{x}+2) - (x^{3/2}+1)]} \end{aligned}$$

and

$$g'_{hI-M_2I}(x) = \frac{3(1-\sqrt{x}) \times k_{10}(x)}{\left(\begin{aligned} &2\sqrt{x}\sqrt{2x+2} \left[- (x^{3/2}+1) + \right. \right. \\ &\left. \left. + \sqrt{2x+2}(2x-3\sqrt{x}+2) \right]^2 \right)}, \end{aligned} \right)$$

where $k_{10}(x) = k_8(x) > 0$. This gives

$$g'_{hI-M_2I}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases}.$$

Also we have

$$\beta = \sup_{x \in (0, \infty)} g_{hI-M_2I}(x) = \lim_{x \rightarrow 1} g_{hI-M_2I}(x) = \frac{4}{3}.$$

12. For $D_{M_2I}^5(P||Q) \leq \frac{1}{8}D_{K_0\Delta}^{36}(P||Q)$: Let us consider

$g_{M_2I-K_0\Delta}(x) = \frac{f_{M_2I}''(x)}{f_{K_0\Delta}''(x)}$, then we have

$$g_{M_2I-K_0\Delta}(x) = \frac{16x(x+1)^2}{3\sqrt{2x+2}(\sqrt{x}-1)^2} \times \frac{\left[\sqrt{2x+2}(2x-3\sqrt{x}+2) - (x^{3/2}+1)\right]}{\left(\frac{3x^4+6x^{7/2}+20x^3+34x^{5/2}}{+66x^2+34x^{3/2}+20x+6\sqrt{x}+3}\right)}$$

and

$$g'_{M_2I-K_0\Delta}(x) = -\frac{8(x+1) \times k_{11}(x)}{\sqrt{2x+2}(\sqrt{x}-1)^3} \times \frac{1}{\left(\frac{3x^4+6x^{7/2}+20x^3+34x^{5/2}}{+66x^2+34x^{3/2}+20x+6\sqrt{x}+3}\right)^2},$$

where

$$k_{11}(x) = \sqrt{2x+2}(\sqrt{x}+1)u(x) - \left(\begin{array}{l} 2x^7 + 2x^{13/2} + 7x^6 + 12x^{11/2} + \\ +11x^5 + 94x^{9/2} + 76x^4 + \\ +104x^{7/2} + 76x^3 + 94x^{5/2} + 11x^2 \\ +12x^{3/2} + 7x + 2\sqrt{x} + 2 \end{array} \right),$$

with

$$u(x) = \left(\begin{array}{l} 4x^6 - 9x^{11/2} + 24x^5 - 41x^{9/2} + \\ +60x^4 + 50x^{7/2} - 48x^3 + 50x^{5/2} + \\ +60x^2 - 41x^{3/2} + 24x - 9\sqrt{x} + 4 \end{array} \right).$$

This gives

$$g'_{M_2I-K_0\Delta}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases},$$

provided $k_{11}(x) > 0$. In order to prove $k_{11}(x) > 0$, let us consider

$$v(t) = u(t^2) = \left(\begin{array}{l} 4t^{12} - 9t^{11} + 24t^{10} - 41t^9 + \\ +60t^8 + 50t^7 - 48t^6 + 50t^5 + \\ +60t^4 - 41t^3 + 24t^2 - 9t + 4 \end{array} \right).$$

Solving the polynomial equation $v(t) = 0$, we observe that there are no real solutions. All the twelve solutions are complex and are given by

$$\begin{aligned} &-0.9437538663 \pm 0.3306488166 I, \\ &-0.3823946004 \pm 2.215272138 I, \\ &-0.07566691909 \pm 0.4383503779 I, \\ &0.3872722043 \pm 0.2946782782 I, \\ &0.5042070498 \pm 0.8635827991 I, \\ &1.635336132 \pm 1.244339331 I. \end{aligned}$$

This means that for all $t > 0$, either $v(t) > 0$ or $v(t) < 0$. Calculating a particular value of $v(t)$, for example for $t = 1$, we get $v(1) = 128 > 0$. This means that $v(t) > 0$,

for all $t > 0$, and hence $u(x) > 0, \forall x > 0$. Let us consider

$$h_{11}(x) = [\sqrt{2x+2}(\sqrt{x}+1)u(x)]^2 - [\sqrt{x}(x+1)]^2 \times \left(\begin{array}{l} 2x^7 + 2x^{13/2} + 7x^6 + 12x^{11/2} + \\ +11x^5 + 94x^{9/2} + 76x^4 + \\ +104x^{7/2} + 76x^3 + 94x^{5/2} + \\ +11x^2 + 12x^{3/2} + 7x + 2\sqrt{x} + 2 \end{array} \right)^2.$$

After simplifications, we have

$$h_{11}(x) = x(x+1)^2(\sqrt{x}-1)^4 \times \left(\begin{array}{l} 8 + 2608x^5 + 218x + 112x^{3/2} + 451x^2 + \\ +24\sqrt{x} + 1910x^3 + 1180x^{5/2} + 5612x^{7/2} + \\ +1420x^{9/2} + 1124x^{11/2} + 1124x^{13/2} + \\ +1910x^9 + 3745x^8 + 3745x^4 + 218x^{11} + \\ +1983x^4(x^2-1)^2 + 1420x^{15/2} + \\ +5612x^{17/2} + 2608x^7 + 451x^{10} + \\ +28x^{12} + 1180x^{19/2} + 24x^{23/2} + 112x^{21/2} \end{array} \right).$$

Since $h_{11}(x) > 0$, this gives that $k_{11}(x) > 0$. Also we have

$$\beta = \sup_{x \in (0, \infty)} g_{M_2I-K_0\Delta}(x) = \lim_{x \rightarrow 1} g_{M_2I-K_0\Delta}(x) = \frac{1}{8}.$$

13. For $D_{M_2I}^5(P||Q) \leq \frac{3}{8}D_{TJ}^{22}(P||Q)$: Let us consider $g_{M_2I-TJ}(x) = f_{M_2I}''(x)/f_{TJ}''(x)$, then we have

$$g_{M_2I-TJ}(x) = \frac{4 \left[(2x-3\sqrt{x}+2)\sqrt{2x+2} - (x^{3/2}+1) \right]}{3\sqrt{2x+2}(x-1)^2}$$

and

$$g'_{M_2I-TJ}(x) = -\frac{2 \times k_{12}(x)}{3\sqrt{2x+2}\sqrt{x}(x+1)(x-1)^3},$$

where

$$k_{12}(x) = 2\sqrt{2x+2} \times \left(\begin{array}{l} x^2(\sqrt{x}-2)^2 + 3x(\sqrt{x}-1)^2 + \\ + (2\sqrt{x}-1)^2 + \sqrt{x}(x^2+1) \end{array} \right) - \left(x^{7/2} + 3x^{5/2} + 4x^{3/2} + 3x + 4x^2 + 1 \right).$$

This gives

$$g'_{M_2I-TJ}(x) \begin{cases} > 0 & x < 1 \\ < 0 & x > 1 \end{cases},$$

provided $k_{12}(x) > 0$. In order to prove $k_{12}(x) > 0$, let us consider

$$h_{12}(x) = (2\sqrt{2x+2})^2 \times \left(\begin{array}{l} x^2(\sqrt{x}-2)^2 + 3x(\sqrt{x}-1)^2 + \\ + (2\sqrt{x}-1)^2 + \sqrt{x}(x^2+1) \end{array} \right)^2 - \left(x^{7/2} + 3x^{5/2} + 4x^{3/2} + 3x + 4x^2 + 1 \right)^2.$$

After simplifications, we have

$$h_{12}(x) = (\sqrt{x} - 1)^4 \times \left(\begin{aligned} &(x+1)(2x^4 + 45x^2 + 2) + \\ &+ 5x^4(\sqrt{x} - 2)^2 + 5(2\sqrt{x} - 1)^2 + \\ &+ x(\sqrt{x} - 1)^2(42x^2 + 65x + 42) \end{aligned} \right).$$

Since $h_{12}(x) > 0$, this gives that $k_{12}(x) > 0$. Also we have

$$\beta = \sup_{x \in (0, \infty)} g_{M_2 I \text{--} T J}(x) = \lim_{x \rightarrow 1} g_{M_2 I \text{--} T J}(x) = \frac{3}{8}.$$

14. For $\mathbf{D}_{M_2 I}^5(\mathbf{P}||\mathbf{Q}) \leq \frac{1}{5}\mathbf{D}_{T M_1}^{26}(\mathbf{P}||\mathbf{Q})$: Let us consider $g_{M_2 I \text{--} T M_1}(x) = f_{M_2 I}''(x)/f_{T M_1}''(x)$, then we have

$$\begin{aligned} g_{M_2 I \text{--} T M_1}(x) &= \\ &= \frac{2\sqrt{x} \left(\begin{aligned} &(2x - 3\sqrt{x} + 2)\sqrt{2x+2} - \\ &- (x^{3/2} + 1) \end{aligned} \right)}{3 \left(\begin{aligned} &\sqrt{2x+2}(x^2 - 2x^{3/2} - 2\sqrt{x} + 1) + \\ &+ 2\sqrt{x}(x^{3/2} + 1) \end{aligned} \right)} \end{aligned}$$

and

$$\begin{aligned} g'_{M_2 I \text{--} T M_1}(x) &= \\ &= - \frac{(\sqrt{x} - 1) \times k_{13}(x)}{\left(\begin{aligned} &3\sqrt{x}(x+1) \left[-2\sqrt{x}(x^{3/2} + 1) + \right. \\ &\left. + \sqrt{2x+2}(x^2 - 2x^{3/2} - 2\sqrt{x} + 1) \right]^2 \end{aligned} \right)}, \end{aligned}$$

where

$$\begin{aligned} k_{13}(x) &= 4(x+1)^2 \times \\ &\times \left[(x^{3/2} + 1)(\sqrt{x} - 1)^2 + 3x(\sqrt{x} + 1) \right] - \\ &- \sqrt{2x+2} \left(\begin{aligned} &x^4 + x^{7/2} + 7x^3 + 5x^{5/2} + \\ &+ 20x^2 + 5x^{3/2} + 7x + \sqrt{x} + 1 \end{aligned} \right). \end{aligned}$$

This gives

$$g'_{M_2 I \text{--} T M_1}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases},$$

provided $k_{13}(x) > 0$. In order to prove $k_{13}(x) > 0$, let us consider

$$\begin{aligned} h_{13}(x) &= [4(x+1)^2]^2 \times \\ &\times \left[(x^{3/2} + 1)(\sqrt{x} - 1)^2 + 3x(\sqrt{x} + 1) \right]^2 - \\ &- \left\{ \sqrt{2x+2} \left(\begin{aligned} &x^4 + x^{7/2} + 7x^3 + 5x^{5/2} + \\ &+ 20x^2 + 5x^{3/2} + 7x + \sqrt{x} + 1 \end{aligned} \right) \right\}^2. \end{aligned}$$

After simplifications, we have

$$\begin{aligned} h_{13}(x) &= 2(x+1)(\sqrt{x} - 1)^4 \times \\ &\times \left(\begin{aligned} &6x^6 + x^5(\sqrt{x} - 3)^2 + 30x^5 + 36x^{9/2} + \\ &+ 60x^4 + 114x^{7/2} + 40x^3 + 114x^{5/2} + \\ &+ 60x^2 + 36x^{3/2} + 30 + (3\sqrt{x} - 1)^2 + 6 \end{aligned} \right). \end{aligned}$$

Since $h_{13}(x) > 0$, this gives that $k_{13}(x) > 0$. Also we have

$$\beta = \sup_{x \in (0, \infty)} g_{M_2 I \text{--} T M_1}(x) = \lim_{x \rightarrow 1} g_{M_2 I \text{--} T M_1}(x) = \frac{1}{5}.$$

15. For $\mathbf{D}_{M_2 I}^5(\mathbf{P}||\mathbf{Q}) \leq \frac{3}{7}\mathbf{D}_{J M_1}^{19}(\mathbf{P}||\mathbf{Q})$: Let us consider $g_{M_2 I \text{--} J M_1}(x) = f_{M_2 I}''(x)/f_{J M_1}''(x)$, then we have

$$\begin{aligned} g_{M_2 I \text{--} J M_1}(x) &= \\ &= \frac{4x \left(\begin{aligned} &(2x - 3\sqrt{x} + 2) \times \\ &\times \sqrt{2x+2} - (x^{3/2} + 1) \end{aligned} \right)}{3 \left(\begin{aligned} &\sqrt{x}\sqrt{2x+2}(x+1) \times \\ &\times (x - 4x + 1) + 4x(x^{3/2} + 1) \end{aligned} \right)} \end{aligned}$$

and

$$\begin{aligned} g'_{M_2 I \text{--} J M_1}(x) &= \\ &= - \frac{2\sqrt{x}(\sqrt{x} - 1) \times k_{14}(x)}{\left(\begin{aligned} &3(x+1) \left[4x(x^{3/2} + 1) + (x+1) \times \right. \\ &\left. \times \sqrt{x}\sqrt{2x+2}(x - 4\sqrt{x} + 1) \right]^2 \end{aligned} \right)}, \end{aligned}$$

where

$$\begin{aligned} k_{14}(x) &= 2\sqrt{2x+2}(\sqrt{x} + 1)(x+1) \times \\ &\times \left[x(\sqrt{x} - 2)^2 + (2\sqrt{x} - 1)^2 + \sqrt{x}(x+1) \right] - \\ &- \left(\begin{aligned} &x^4 + x^{7/2} + 11x^3 + 3x^{5/2} + \\ &+ 32x^2 + 3x^{3/2} + 11x + \sqrt{x} + 1 \end{aligned} \right). \end{aligned}$$

This gives

$$g'_{M_2 I \text{--} J M_1}(x) \begin{cases} > 0 & x < 1 \\ < 0 & x > 1 \end{cases},$$

provided $k_{14}(x) > 0$. In order to prove $k_{14}(x) > 0$, let us consider

$$\begin{aligned} h_{14}(x) &= \left\{ \begin{aligned} &2\sqrt{2x+2}(\sqrt{x} + 1)(x+1) \times \\ &\times \left[\begin{aligned} &x(\sqrt{x} - 2)^2 + (2\sqrt{x} - 1)^2 \\ &+ \sqrt{x}(x+1) \end{aligned} \right] \end{aligned} \right\}^2 - \\ &- \left(\begin{aligned} &x^4 + x^{7/2} + 11x^3 + 3x^{5/2} + \\ &+ 32x^2 + 3x^{3/2} + 11x + \sqrt{x} + 1 \end{aligned} \right)^2. \end{aligned}$$

After simplifications, we have

$$\begin{aligned} h_{14}(x) &= (\sqrt{x} - 1)^2 \times \\ &\times \left(\begin{aligned} &(3x^5 + 2x^{5/2} + 3)(\sqrt{x} - 1)^2 + 48x^{9/2} + \\ &+ 136x^{7/2} + 136x^{5/2} + 48x^{3/2} + 4x^6 + \\ &+ 44x^5 + 56x^4 + 56x^2 + 44x + 4 \end{aligned} \right). \end{aligned}$$

Since $h_{14}(x) > 0$, this gives that $k_{14}(x) > 0$. Also we have

$$\beta = \sup_{x \in (0, \infty)} g_{M_2 I \text{--} J M_1}(x) = \lim_{x \rightarrow 1} g_{M_2 I \text{--} J M_1}(x) = \frac{3}{7}.$$

16. For $D_{M_2I}^5(\mathbf{P}||\mathbf{Q}) \leq 3D_{M_1I}^2(\mathbf{P}||\mathbf{Q})$: Let us consider $g_{M_2I-M_1I}(x) = f''_{M_2I}(x)/f''_{M_1I}(x)$, then we have

$$g_{M_2I-M_1I}(x) = \frac{\sqrt{2x+2}(2x-3\sqrt{x}+2) - (x^{3/2}+1)}{3[\sqrt{2x+2}(x-\sqrt{x}+1) - (x^{3/2}+1)]}$$

and

$$g'_{M_2I-M_1I}(x) = -\frac{(\sqrt{x}-1) \times k_{15}(x)}{\left(\frac{3\sqrt{x}\sqrt{2x+2}[-(x^{3/2}+1)+\sqrt{2x+2}(x-\sqrt{x}+1)]^2}{+ \sqrt{2x+2}(x-\sqrt{x}+1)} \right)},$$

Where $k_{15}(x) = k_8(x) > 0$. This give

$$g'_{M_2I-M_1I}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases}.$$

Also we have

$$\beta = \sup_{x \in (0, \infty)} g_{M_2I-M_1I}(x) = \lim_{x \rightarrow 1} g_{M_2I-M_1I}(x) = 3.$$

17. For $D_{TJ}^{22}(\mathbf{P}||\mathbf{Q}) \leq \frac{8}{13}D_{TM_2}^{25}(\mathbf{P}||\mathbf{Q})$: Let us consider $g_{TJ-TM_2}(x) = f''_{TJ}(x)/f''_{TM_2}(x)$, then we have

$$g_{TJ-TM_2}(x) = \frac{3(x-1)^2\sqrt{2x+2}}{2 \left(\frac{2\sqrt{x}(x^{3/2}+1) + \sqrt{2x+2} \times (3x^2-4x^{3/2}-4\sqrt{x}+3)}{+ \sqrt{2x+2}(3x^2-4x^{3/2}-4\sqrt{x}+3)} \right)}$$

and

$$g'_{TJ-TM_2}(x) = -\frac{3(x-1) \times k_{16}(x)}{\left(\frac{\sqrt{x}\sqrt{2x+2}[2\sqrt{x}(x^{3/2}+1) + \sqrt{2x+2}(3x^2-4x^{3/2}-4\sqrt{x}+3)]^2}{+ \sqrt{2x+2}(3x^2-4x^{3/2}-4\sqrt{x}+3)} \right)},$$

where

$$k_{16}(x) = 2(x+1)\sqrt{2x+2} \times \left[(\sqrt{x}-1)^4 + \sqrt{x}(x+1) \right] - (\sqrt{x}+1) \left(\frac{x^2(\sqrt{x}-2)^2 + (2\sqrt{x}-1)^2}{+ 3\sqrt{x}(x^2+1)} \right).$$

This give

$$g'_{TJ-TM_2}(x) \begin{cases} > 0 & x < 1 \\ < 0 & x > 1 \end{cases},$$

provided $k_{16}(x) > 0$. In order to prove $k_{16}(x) > 0$, let us consider

$$h_{16}(x) = (2(x+1)\sqrt{2x+2})^2 \times \left[(\sqrt{x}-1)^4 + \sqrt{x}(x+1) \right] - (\sqrt{x}+1)^2 \times \left(\frac{x^2(\sqrt{x}-2)^2 + (2\sqrt{x}-1)^2}{+ 3\sqrt{x}(x^2+1)} \right)^2.$$

After simplifications, we have

$$h_{16}(x) = (\sqrt{x}-1)^4 \times \left(\frac{x(42x^2+65x+42)(\sqrt{x}-1)^2 + 5x^4(\sqrt{x}-2)^2 + 5(2\sqrt{x}-1)^2}{+ (x+1)(2x^4+45x^2+2)} \right).$$

Since $h_{16}(x) > 0$, this gives that $k_{16}(x) > 0$. Also we have

$$\beta = \sup_{x \in (0, \infty)} g_{TJ-TM_2}(x) = \lim_{x \rightarrow 1} g_{TJ-TM_2}(x) = \frac{8}{13}.$$

18. For $D_{TM_1}^{26}(\mathbf{P}||\mathbf{Q}) \leq \frac{15}{13}D_{TM_2}^{25}(\mathbf{P}||\mathbf{Q})$: Let us consider $g_{TM_1-TM_2}(x) = f''_{TM_1}(x)/f''_{TM_2}(x)$, then we have

$$g_{TM_1-TM_2}(x) = \frac{\left(\frac{3\sqrt{2x+2}(x^2-2x^{3/2}-2\sqrt{x}+1) + 2\sqrt{x}(x^{3/2}+1)}{\sqrt{2x+2}(3x^2-4x^{3/2}-4\sqrt{x}+3) + 2\sqrt{x}(x^{3/2}+1)} \right)}{\left(\frac{3\sqrt{2x+2}(x^2-2x^{3/2}-2\sqrt{x}+1) + 2\sqrt{x}(x^{3/2}+1)}{\sqrt{2x+2}(3x^2-4x^{3/2}-4\sqrt{x}+3) + 2\sqrt{x}(x^{3/2}+1)} \right)}$$

and

$$g'_{TM_1-TM_2}(x) = -\frac{6(\sqrt{x}-1)k_{17}(x)}{\left(\frac{\sqrt{x}\sqrt{2x+2}[2\sqrt{x}(x^{3/2}+1) + \sqrt{2x+2}(3x^2-4x^{3/2}-4\sqrt{x}+3)]^2}{+ \sqrt{2x+2}(3x^2-4x^{3/2}-4\sqrt{x}+3)} \right)},$$

where

$$k_{17}(x) = - \left(\frac{\sqrt{2x+2}(\sqrt{x}+1) \times (x+1)(x^2+4x+1)}{+ 2 \left(\frac{x^4+x^{7/2}+x^3+8x^{5/2}}{+ 2x^2+8x^{3/2}+x+\sqrt{x}+1} \right)} \right).$$

This give

$$g'_{TM_1-TM_2}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases},$$

provided $k_{17}(x) > 0$. In order to show $k_{17}(x) > 0$, let us consider

$$h_{17}(x) = - \left(\frac{\sqrt{2x+2}(\sqrt{x}+1) \times (x+1)(x^2+4x+1)}{+ 2 \left(\frac{x^4+x^{7/2}+x^3+8x^{5/2}}{+ 2x^2+8x^{3/2}+x+\sqrt{x}+1} \right)} \right)^2.$$

After simplifications, we have

$$h_{17}(x) = 2(\sqrt{x}-1)^4 \times \left(\frac{1+6\sqrt{x}+30x^{3/2}+72x^{5/2}+12x^5}{+ 57x^2+94x^3+57x^4+x^6+12x+6x^{11/2}+72x^{7/2}+30x^{9/2}} \right).$$

Since $h_{17}(x) > 0$, proving that $k_{17}(x) > 0$. Also we have

$$\beta = \sup_{x \in (0, \infty)} g_{TM_1-TM_2}(x) = \lim_{x \rightarrow 1} g_{TM_1-TM_2}(x) = \frac{15}{13}.$$

19. **For $D_{TM_2}^{25}(P||Q) \leq \frac{13}{12}D_{Th}^{24}(P||Q)$:** Let us consider $g_{TM_2-Th}(x) = f''_{TM_2}(x)/f''_{Th}(x)$, then we have

$$g_{TM_2-Th}(x) = \frac{\left(\sqrt{2x+2} \left(3x^2 - 4x^{3/2} - 4\sqrt{x} + 3 \right) + 2\sqrt{x} \left(x^{3/2} + 1 \right) \right)}{3\sqrt{2x+2} (x + \sqrt{x} + 1) (\sqrt{x} - 1)^2}$$

and

$$g'_{TM_2-Th}(x) = - \frac{k_{18}(x)}{\left(\frac{6(\sqrt{x}-1)^3 \sqrt{x}(x+1) \times}{\times \sqrt{2x+2} (x + \sqrt{x} + 1)^2} \right)},$$

where $k_{18}(x) = k_{17}(x) > 0$. This gives

$$g'_{TM_2-Th}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases}.$$

Also we have

$$\beta = \sup_{x \in (0, \infty)} g_{TM_2-Th}(x) = \lim_{x \rightarrow 1} g_{TM_2-Th}(x) = \frac{13}{12}.$$

20. **For $D_{Th}^{24}(P||Q) \leq \frac{4}{3}D_{TM_3}^{23}(P||Q)$:** Let us consider $g_{Th-TM_3}(x) = f''_{Th}(x)/f''_{TM_3}(x)$, then we have

$$g_{Th-TM_3}(x) = \frac{(x + \sqrt{x} + 1) (\sqrt{x} - 1)^2 \sqrt{2x+2}}{(x^2 + 1) \sqrt{2x+2} - 2\sqrt{x} (x^{3/2} + 1)}$$

and

$$g'_{Th-TM_3}(x) = - \frac{(\sqrt{x} - 1) \times k_{19}(x)}{\left(\frac{\sqrt{x}\sqrt{2x+2} [(x^2+1)\sqrt{2x+2} - 2\sqrt{x}(x^{3/2}+1)]^2}{-2\sqrt{x}(x^{3/2}+1)} \right)},$$

where $k_{19}(x) = k_{17}(x) > 0$. This gives

$$g'_{Th-TM_3}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases}.$$

Also we have

$$\beta = \sup_{x \in (0, \infty)} g_{Th-TM_3}(x) = \lim_{x \rightarrow 1} g_{Th-TM_3}(x) = \frac{4}{3}.$$

21. **For $D_{Th}^{24}(P||Q) \leq \frac{12}{5}D_{JM_2}^{18}(P||Q)$:** Let us consider $g_{Th-JM_2}(x) = f''_{Th}(x)/f''_{JM_2}(x)$, then we have

$$g_{Th-JM_2}(x) = \frac{6(x + \sqrt{x} + 1) (\sqrt{x} - 1)^2 \sqrt{x}\sqrt{2x+2}}{\left(\frac{\sqrt{2x+2}\sqrt{x}(x+1) \times}{\times (3x - 8\sqrt{x} + 3) + 4x(x^{3/2} + 1)} \right)}$$

and

$$g'_{Th-JM_2}(x) = - \frac{3(\sqrt{x} - 1) \sqrt{x}\sqrt{2x+2} \times k_{20}(x)}{\left(\frac{(x+1) [\sqrt{2x+2}\sqrt{x}(x+1) \times]}{\times (3x - 8\sqrt{x} + 3) + 4x(x^{3/2} + 1)} \right)^2},$$

where

$$k_{20}(x) = \sqrt{2x+2} (\sqrt{x} + 1) (x + 1) \times \left(\frac{x(\sqrt{x}-2)^2 + (2\sqrt{x}-1)^2 +}{+4(x+1)(\sqrt{x}-1)^2 + 10x} \right) - 4 \left(\frac{x^4 + x^{7/2} + x^3 + 8x^{5/2} +}{+2x^2 + 8x^{3/2} + x + \sqrt{x} + 1} \right).$$

This gives

$$g'_{Th-JM_2}(x) \begin{cases} > 0 & x < 1 \\ < 0 & x > 1 \end{cases},$$

provided $k_{20}(x) > 0$. In order to show $k_{20}(x) > 0$, let us consider

$$h_{20}(x) = [\sqrt{2x+2} (\sqrt{x} + 1) (x + 1)]^2 \times \left(\frac{x(\sqrt{x}-2)^2 + (2\sqrt{x}-1)^2 +}{+4(x+1)(\sqrt{x}-1)^2 + 10x} \right)^2 - \left[4 \left(\frac{x^4 + x^{7/2} + x^3 + 8x^{5/2} +}{+2x^2 + 8x^{3/2} + x + \sqrt{x} + 1} \right) \right]^2.$$

After simplifications, we have

$$h_{20}(x) = 2(\sqrt{x} - 1)^4 \times \left(\frac{(\sqrt{x} - 1)^2 (9 + 14x^{5/2} + 9x^5) + 57x +}{+118x^{7/2} + 8x^6 + 8 + 118x^{5/2} + 57x^5 +}{+39x^2 + 39x^4 + 30x^{3/2} + 30x^{9/2}} \right).$$

Since $h_{20}(x) > 0$, proving that $k_{20}(x) > 0$. Also we have

$$\beta = \sup_{x \in (0, \infty)} g_{Th-JM_2}(x) = \lim_{x \rightarrow 1} g_{Th-JM_2}(x) = \frac{12}{5}.$$

22. **For $D_{JM_2}^{18}(P||Q) \leq \frac{5}{4}D_{Jh}^{17}(P||Q)$:** Let us consider $g_{JM_2-Jh}(x) = f''_{JM_2}(x)/f''_{Jh}(x)$, then we have

$$g_{JM_2-Jh}(x) = \frac{\left(\frac{\sqrt{2x+2}\sqrt{x}(x+1) \times}{\times (3x - 8\sqrt{x} + 3) + 4x(x^{3/2} + 1)} \right)}{3\sqrt{2x+2}\sqrt{x}(x+1)(\sqrt{x}-1)^2}$$

and

$$g'_{JM_2-Jh}(x) = -\frac{k_{21}(x)}{3\sqrt{2x+2}\sqrt{x}(x+1)(\sqrt{x}-1)^3},$$

where

$$k_{21}(x) = 2 \left[x^3 + \sqrt{x}(x+1)(\sqrt{x}-1)^2 \right] + 2 \left(6x^{3/2} + 1 \right) - \sqrt{2x+2}(\sqrt{x}+1)(x+1)^2.$$

This gives

$$g'_{JM_2-Jh}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases}.$$

provided $k_{21}(x) > 0$. In order to prove $k_{21}(x) > 0$, let us consider

$$h_{21}(x) = 4 \left(\frac{\sqrt{x}(x+1)(\sqrt{x}-1)^2 +}{x^3 + 2(6x^{3/2} + 1)} \right)^2 - [\sqrt{2x+2}(\sqrt{x}+1)(x+1)^2]^2.$$

After simplifications, we have

$$h_{21}(x) = 2(\sqrt{x}-1)^4 \times \left(\frac{x^4 + 6x^{7/2} + 6x^3 + 6x^{5/2} +}{+28x^2 + 6x^{3/2} + 6x + 6\sqrt{x} + 1} \right).$$

Since $h_{21}(x) > 0$, this gives that $k_{21}(x) > 0$. Also we have

$$\beta = \sup_{x \in (0, \infty)} g_{JM_2-Jh}(x) = \lim_{x \rightarrow 1} g_{JM_2-Jh}(x) = \frac{5}{4}.$$

23. For $D_{K_0\Delta}^{36}(\mathbf{P}||\mathbf{Q}) \leq \frac{3}{2}D_{K_0I}^{35}(\mathbf{P}||\mathbf{Q})$: Let us consider $g_{K_0\Delta-K_0I}(x) = f''_{K_0\Delta}(x)/f''_{K_0I}(x)$, then we have

$$g_{K_0\Delta-K_0I}(x) = \frac{\left(\frac{3x^4 + 6x^{7/2} + 20x^3 + 34x^{5/2} +}{+66x^2 + 34x^{3/2} + 20x + 6\sqrt{x} + 3} \right)}{(x+1)^2(3x^2 + 6x^{3/2} + 14x + 6\sqrt{x} + 3)}$$

and

$$g'_{K_0\Delta-K_0I}(x) = -\frac{8(x-1)\sqrt{x}(3x+8\sqrt{x}+3)}{(x+1)^3} \times \frac{(3x^2 + 4x^{3/2} + 10x + 4\sqrt{x} + 3)}{(3x^2 + 6x^{3/2} + 14x + 6\sqrt{x} + 3)^2}.$$

This gives

$$g'_{K_0\Delta-K_0I}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases}.$$

Also we have

$$\beta = \sup_{x \in (0, \infty)} g_{K_0\Delta-K_0I}(x) = \lim_{x \rightarrow 1} g_{K_0\Delta-K_0I}(x) = \frac{3}{2}.$$

24. For $D_{TM_3}^{23}(\mathbf{P}||\mathbf{Q}) \leq \frac{9}{16}D_{K_0I}^{35}(\mathbf{P}||\mathbf{Q})$: Let us consider $g_{TM_3-K_0I}(x) = f''_{TM_3}(x)/f''_{K_0I}(x)$, then we have

$$g_{TM_3-K_0I}(x) = \frac{\left[\sqrt{2x+2}(x^2+1) - 2\sqrt{x}(x^{3/2}+1) \right]}{\sqrt{2x+2}(\sqrt{x}-1)^2} \times \frac{8\sqrt{x}}{(3x^2 + 6x^{3/2} + 14x + 6\sqrt{x} + 3)}$$

and

$$g'_{TM_3-K_0I}(x) = -\frac{4 \times k_{22}(x)}{\left(\frac{\sqrt{x}\sqrt{2x+2}(x+1)(\sqrt{x}-1)^3 \times}{\times (3x^2 + 6x^{3/2} + 14x + 6\sqrt{x} + 3)^2} \right)},$$

where

$$k_{22}(x) = \sqrt{2x+2} \times \left(\frac{3x^{11/2} + 28x^2 + 3\sqrt{x} + 33x^{3/2} +}{+60x^3 + 33x^4 + 3x^5 + 60x^{5/2} +}{+28x^{7/2} + x^{9/2} + x + 3} \right) - 2\sqrt{x} \left(\frac{6\sqrt{x} + 40x^{3/2} + 40x^{7/2} +}{+9x + 84x^{5/2} + 6x^5 + 9x^4 +}{+25x^3 + 6x^{9/2} + 25x^2 + 6} \right).$$

This gives

$$g'_{TM_3-K_0I}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases},$$

provided $k_{22}(x) > 0$. In order to prove $k_{22}(x) > 0$, let us consider

$$h_{22}(x) = (\sqrt{2x+2})^2 \times \left(\frac{3x^{11/2} + 28x^2 + 3\sqrt{x} + 33x^{3/2} +}{+60x^3 + 33x^4 + 3x^5 + 60x^{5/2} +}{+28x^{7/2} + x^{9/2} + x + 3} \right)^2 - \left[2\sqrt{x} \left(\frac{6\sqrt{x} + 40x^{3/2} + 40x^{7/2} +}{+9x + 84x^{5/2} + 6x^5 + 9x^4 +}{+25x^3 + 6x^{9/2} + 25x^2 + 6} \right) \right]^2.$$

After simplification, we have

$$h_{22}(x) = 2(\sqrt{x}-1)^4 \times \left(\frac{9 + 54\sqrt{x} + 1326x^4 + 54x^{19/2} +}{+246x^{17/2} + 1324x^{9/2} + 1324x^{11/2} +}{+952x^{15/2} + 1808x^{13/2} + 1582x^3 +}{+952x^{5/2} + 114x + 601x^2 + 352x^5 +}{+9x^{10} + 114x^9 + 601x^8 + 1582x^7 +}{+1326x^6 + 1808x^{7/2} + 246x^{3/2}} \right).$$

Since $h_{22}(x) > 0$, proving that $k_{22}(x) > 0$. Also we have

$$\beta = \sup_{x \in (0, \infty)} g_{TM_3-K_0I}(x) = \lim_{x \rightarrow 1} g_{TM_3-K_0I}(x) = \frac{9}{16}.$$

25. For $D_{M_1 I}^2(\mathbf{P}||\mathbf{Q}) \leq \frac{1}{16} D_{K_0 I}^{35}(\mathbf{P}||\mathbf{Q})$: Let us consider $g_{M_1 I-K_0 I}(x) = f_{M_1 I}''(x)/f_{K_0 I}''(x)$, then we have

$$g_{M_1 I-K_0 I}(x) = \frac{16x \left[\sqrt{2x+2} (x - \sqrt{x} + 1) - (x^{3/2} + 1) \right]}{\left(\sqrt{2x+2} (\sqrt{x} - 1)^2 \times \left(3x^2 + 6x^{3/2} + 14x + 6\sqrt{x} + 3 \right) \right)}$$

and

$$g_{M_1 I-K_0 I}'(x) = - \frac{8 \times k_{23}(x)}{\left((\sqrt{x} - 1)^3 \sqrt{2x+2} (x+1) \times \left(3x^2 + 6x^{3/2} + 14x + 6\sqrt{x} + 3 \right)^2 \right)},$$

where

$$k_{23}(x) = \sqrt{2x+2} (x+1) (\sqrt{x} + 1) \times \left(\frac{6(x^2+1)(\sqrt{x}-1)^2 + 2x^{3/2}}{+3x^2(\sqrt{x}+4) + 3\sqrt{x}(4\sqrt{x}+1)} \right) - \left(\frac{6x^{9/2} + 40x^{7/2} + 84x^{5/2} + 40x^{3/2} + 6\sqrt{x}}{+6x^5 + 9x^4 + 25x^3 + 25x^2 + 9x + 6} \right).$$

This gives

$$g_{M_1 I-K_0 I}'(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases},$$

provided $k_{23}(x) > 0$. In order to prove $k_{23}(x) > 0$, let us consider

$$h_{23}(x) = \left[\sqrt{2x+2} (x+1) (\sqrt{x} + 1) \right]^2 \times \left(\frac{6(x^2+1)(\sqrt{x}-1)^2 + 2x^{3/2}}{+3x^2(\sqrt{x}+4) + 3\sqrt{x}(4\sqrt{x}+1)} \right)^2 - \left(\frac{6x^{9/2} + 40x^{7/2} + 84x^{5/2} + 40x^{3/2} + 6\sqrt{x}}{+6x^5 + 9x^4 + 25x^3 + 25x^2 + 9x + 6} \right)^2.$$

After simplification, we get

$$h_{23}(x) = (\sqrt{x} - 1)^6 \times \left(\frac{36 + 72\sqrt{x} + 198x + 1314x^{5/2} + 396x^{3/2} + 1314x^{9/2} + 396x^{11/2} + 1717x^3 + 765x^2 + 2012x^{7/2} + 72x^{13/2} + 765x^5 + 1717x^4 + 36x^7 + 198x^6}{+36x^7 + 198x^6} \right).$$

Since $h_{23}(x) > 0$, this gives that $k_{23}(x) > 0$ Also we have

$$\beta = \sup_{x \in (0, \infty)} g_{M_1 I-K_0 I}(x) = \lim_{x \rightarrow 1} g_{M_1 I-K_0 I}(x) = \frac{1}{16}.$$

26. For $D_{Jh}^{17}(\mathbf{P}||\mathbf{Q}) \leq \frac{1}{4} D_{K_0 I}^{35}(\mathbf{P}||\mathbf{Q})$: Let us consider $g_{Jh-K_0 I}(x) = f_{Jh}''(x)/f_{K_0 I}''(x)$, then we have

$$g_{Jh-K_0 I}(x) = \frac{4\sqrt{x}(x+1)}{3x^2 + 6x^{3/2} + 14x + 6\sqrt{x} + 3}$$

$$g_{Jh-K_0 I}'(x) = - \frac{2(x-1) [2(x^2+1) + (x-1)^2]}{(3x^2 + 6x^{3/2} + 14x + 6\sqrt{x} + 3)^2}.$$

This gives

$$g_{Jh-K_0 I}'(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases}.$$

Also we have

$$\beta = \sup_{x \in (0, \infty)} g_{Jh-K_0 I}(x) = \lim_{x \rightarrow 1} g_{Jh-K_0 I}(x) = \frac{1}{4}.$$

27. For $D_{K_0 I}^{35}(\mathbf{P}||\mathbf{Q}) \leq \frac{16}{15} D_{K_0 M_1}^{34}(\mathbf{P}||\mathbf{Q})$: Let us consider $g_{K_0 I-K_0 M_1}(x) = f_{K_0 I}''(x)/f_{K_0 M_1}''(x)$, then we have

$$g_{K_0 I-K_0 M_1}(x) = \frac{\left(\sqrt{2x+2} (\sqrt{x} - 1)^2 \times \left(3x^2 + 6x^{3/2} + 14x + 6\sqrt{x} + 3 \right) \right)}{\left(\sqrt{2x+2} (x+1) \times \left(3x^2 - 14x + 3 \right) + 16x (x^{3/2} + 1) \right)}$$

and

$$g_{K_0 I-K_0 M_1}'(x) = - \frac{16(\sqrt{x} - 1) \times k_{24}(x)}{\left(\sqrt{x} \sqrt{2x+2} \left[16x (x^{3/2} + 1) + \sqrt{2x+2} (x+1) (3x^2 - 14x + 3) \right]^2 \right)},$$

where $k_{24}(x) = k_{23}(x) > 0$. This gives

$$g_{K_0 I-K_0 M_1}'(x) \begin{cases} > 0 & x < 1 \\ < 0 & x > 1 \end{cases}.$$

Also we have

$$\beta = \sup_{x \in (0, \infty)} g_{K_0 I-K_0 M_1}(x) = \lim_{x \rightarrow 1} g_{K_0 I-K_0 M_1}(x) = \frac{16}{15}.$$

28. For $D_{K_0 M_1}^{34}(\mathbf{P}||\mathbf{Q}) \leq \frac{15}{13} D_{K_0 M_2}^{33}(\mathbf{P}||\mathbf{Q})$: Let us consider $g_{K_0 M_1-K_0 M_2}(x) = f_{K_0 M_1}''(x)/f_{K_0 M_2}''(x)$, then we have

$$g_{K_0 M_1-K_0 M_2}(x) = \frac{3 \left(\sqrt{2x+2} (x+1) \times \left(3x^2 - 14x + 3 \right) + 16x (x^{3/2} + 1) \right)}{\left(\sqrt{2x+2} (x+1) \times \left(9x^2 - 26x + 9 \right) + 16x (x^{3/2} + 1) \right)}$$

and

$$g'_{K_0 M_1 - K_0 M_2}(x) = -\frac{288(x-1)(x+1) \times k_{25}(x)}{\left(\frac{\sqrt{2x+2}(x+1)(9x^2-26x+9)}{+16x(x^{3/2}+1)} \right)^2},$$

where

$$k_{25}(x) = \left(2x^{7/2} + x^{5/2} + 5x^2 + 5x^{3/2} + x + 2 \right) - \sqrt{2x+2}(x+1)^3.$$

This gives

$$g'_{K_0 M_1 - K_0 M_2}(x) \begin{cases} > 0 & x < 1 \\ < 0 & x > 1 \end{cases},$$

provided $k_{25}(x) > 0$. In order to prove $k_{25}(x) > 0$, let us consider a function

$$h_{25}(x) = \left(\frac{2x^{7/2} + x^{5/2} + 5x^2 + 5x^{3/2} + x + 2}{+5x^{3/2} + x + 2} \right)^2 - [\sqrt{2x+2}(x+1)^3]^2.$$

After simplifications, we get

$$h_{25}(x) = (\sqrt{x}-1)^4 \times \left(\frac{2x^5 + 8x^{9/2} + 10x^4 + 20x^{7/2} + 29x^3 + 42x^{5/2} + 29x^2 + 20x^{3/2} + 8\sqrt{x} + 10x + 2}{+20x^{3/2} + 8\sqrt{x} + 10x + 2} \right).$$

Since $h_{25}(x) > 0$, this gives that $k_{25}(x) > 0$. Also we have

$$\beta = \sup_{x \in (0, \infty)} g_{K_0 M_1 - K_0 M_2}(x) = \lim_{x \rightarrow 1} g_{K_0 M_1 - K_0 M_2}(x) = \frac{15}{13}.$$

29. For $\mathbf{D}_{K_0 M_2}^{33}(\mathbf{P}||\mathbf{Q}) \leq \frac{13}{12} \mathbf{D}_{K_0 h}^{32}(\mathbf{P}||\mathbf{Q})$: Let us consider $g_{K_0 M_2 - K_0 h}(x) = f''_{K_0 M_2}(x)/f''_{K_0 h}(x)$, then we have

$$g_{K_0 M_2 - K_0 h}(x) = \frac{\left(\frac{\sqrt{2x+2}(x+1) \times (9x^2-26x+9) + 16x(x^{3/2}+1)}{9\sqrt{2x+2}(x+1)(x-1)^2} \right)}{9\sqrt{2x+2}(x+1)(x-1)^2}$$

and

$$g'_{K_0 M_2 - K_0 h}(x) = -\frac{8 \times k_{26}(x)}{9(x-1)^3(x+1)^2\sqrt{2x+2}},$$

where $k_{26}(x) = k_{25}(x) > 0$ This gives

$$g'_{K_0 M_2 - K_0 h}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases}.$$

Also, we have

$$\beta = \sup_{x \in (0, \infty)} g_{K_0 M_2 - K_0 h}(x) = \lim_{x \rightarrow 1} g_{K_0 M_2 - K_0 h}(x) = \frac{13}{12}.$$

30. For $\mathbf{D}_{K_0 h}^{32}(\mathbf{P}||\mathbf{Q}) \leq \frac{4}{3} \mathbf{D}_{K_0 M_3}^{31}(\mathbf{P}||\mathbf{Q})$: Let us consider $g_{K_0 h - K_0 M_3}(x) = f''_{K_0 h}(x)/f''_{K_0 M_3}(x)$, then we have

$$g_{K_0 h - K_0 M_3}(x) = \frac{3(x-1)^2(x+1)\sqrt{2x+2}}{\left(\frac{\sqrt{2x+2}(x+1) \times (3x^2+2x+3) - 16x(x^{3/2}+1)}{\times (3x^2+2x+3) - 16x(x^{3/2}+1)} \right)}$$

and

$$g'_{K_0 h - K_0 M_3}(x) = -\frac{24(x-1)\sqrt{2x+2} \times k_{27}(x)}{\left(\frac{[\sqrt{2x+2}(x+1) \times (3x^2+2x+3) - 16x(x^{3/2}+1)]^2}{\times (3x^2+2x+3) - 16x(x^{3/2}+1)} \right)},$$

where $k_{27}(x) = k_{25}(x) > 0$ This gives

$$g'_{K_0 h - K_0 M_3}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases},$$

Also we have

$$\beta = \sup_{x \in (0, \infty)} g_{K_0 h - K_0 M_3}(x) = \lim_{x \rightarrow 1} g_{K_0 h - K_0 M_3}(x) = \frac{4}{3}.$$

31. For $\mathbf{D}_{K_0 h}^{32}(\mathbf{P}||\mathbf{Q}) \leq \frac{3}{2} \mathbf{D}_{K_0 J}^{30}(\mathbf{P}||\mathbf{Q})$: Let us consider $g_{K_0 h - K_0 J}(x) = f''_{K_0 h}(x)/f''_{K_0 J}(x)$, then we have

$$g_{K_0 h - K_0 J}(x) = \frac{3(\sqrt{x}+1)^2}{3x+2\sqrt{x}+3},$$

$$g'_{K_0 h - K_0 J}(x) = -\frac{6(x-1)}{\sqrt{x}(3x+2\sqrt{x}+3)^2} \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases}$$

and

$$\beta = \sup_{x \in (0, \infty)} g_{K_0 h - K_0 J}(x) = \lim_{x \rightarrow 1} g_{K_0 h - K_0 J}(x) = \frac{3}{2}.$$

32. For $\mathbf{D}_{K_0 h}^{32}(\mathbf{P}||\mathbf{Q}) \leq \frac{1}{4} \mathbf{D}_{\Psi \Delta}^{45}(\mathbf{P}||\mathbf{Q})$: Let us consider $g_{K_0 h - \Psi \Delta}(x) = f''_{K_0 h}(x)/f''_{\Psi \Delta}(x)$, then we have

$$g_{K_0 h - \Psi \Delta}(x) = \frac{3\sqrt{x}(x+1)^3}{4(x^4+5x^3+12x^2+5x+1)},$$

$$g'_{K_0 h - \Psi \Delta}(x) = -\frac{\left(\frac{8(x-1)^3(x+1)^2 \times (x^2+5x+1)}{\times (x^2+5x+1)} \right)}{3\sqrt{x} \left(\frac{x^4+5x^3+12x^2+5x+1}{+12x^2+5x+1} \right)^2} \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases}$$

and

$$\beta = \sup_{x \in (0, \infty)} g_{K_0 h - \Psi \Delta}(x) = \lim_{x \rightarrow 1} g_{K_0 h - \Psi \Delta}(x) = \frac{1}{4}.$$

33. For $D_{K_0J}^{30}(P||Q) \leq \frac{1}{5}D_{\Psi I}^{44}(P||Q)$: Let us consider $g_{K_0J-\Psi I}(x) = f''_{K_0J}(x)/f''_{\Psi I}(x)$, then we have

$$g_{K_0J-\Psi I}(x) = \frac{\sqrt{x}(3x+2\sqrt{x}+3)(x+1)}{4(x^2+3x+1)(\sqrt{x}+1)^2}$$

and

$$g'_{K_0J-\Psi I}(x) = -\frac{(\sqrt{x}-1)(3x+\sqrt{x}+3)}{8\sqrt{x}(x^2+3x+1)^2(\sqrt{x}+1)^3} \times \left(\begin{aligned} & (x^{5/2}+1)(\sqrt{x}+1) + \\ & +2x(x+1)+x(\sqrt{x}-1)^2 \end{aligned} \right).$$

This gives

$$g'_{K_0J-\Psi I}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases}.$$

Also we have

$$\beta = \sup_{x \in (0, \infty)} g_{K_0J-\Psi I}(x) = \lim_{x \rightarrow 1} g_{K_0J-\Psi I}(x) = \frac{1}{5}.$$

34. For $D_{\Psi \Delta}^{45}(P||Q) \leq \frac{6}{5}D_{\Psi I}^{44}(P||Q)$: It is true in view of (2).

35. For $D_{K_0M_3}^{31}(P||Q) \leq \frac{3}{13}D_{\Psi M_1}^{43}(P||Q)$: Let us consider $g_{K_0M_3-\Psi M_1}(x) = f''_{K_0M_3}(x)/f''_{\Psi M_1}(x)$, then we have

$$g_{K_0M_3-\Psi M_1}(x) = \frac{\sqrt{x} \left(\begin{aligned} & \sqrt{2x+2}(x+1) \times \\ & \times (3x^2+2x+3) - 16x(x^{3/2}+1) \end{aligned} \right)}{4 \left(\begin{aligned} & \sqrt{2x+2}(x+1) \times \\ & \times (x^3-4x^{3/2}+1) + 4x^{3/2}(x^{3/2}+1) \end{aligned} \right)}$$

and

$$g'_{K_0M_3-\Psi M_1}(x) = -\frac{3\sqrt{x}\sqrt{2x+2}(\sqrt{x}-1) \times k_{28}(x)}{\left(\begin{aligned} & 8\sqrt{x}(\sqrt{x}-1)^3(x+1)^2\sqrt{2x+2} \times \\ & \times (4x^2+5x^{3/2}+6x+5\sqrt{x}+4)^2 \end{aligned} \right)},$$

where

$$k_{28}(x) = \sqrt{2x+2}(\sqrt{x}+1) \times \begin{aligned} & \times (x+1)^2 \left(\begin{aligned} & x^4+3x^3+8x^{5/2}+ \\ & +8x^2+8x^{3/2}+3x+1 \end{aligned} \right) - \\ & - 4x \left(\begin{aligned} & 4x^5+6x^{9/2}+6x^4+13x^{7/2}+ \\ & +18x^3+34x^{5/2}+18x^2+ \\ & +13x^{3/2}+6x+6\sqrt{x}+4 \end{aligned} \right). \end{aligned}$$

This gives

$$g'_{K_0M_3-\Psi M_1}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases},$$

provided $k_{28}(x) > 0$. In order to prove $k_{28}(x) > 0$, let us consider

$$h_{28}(x) = [\sqrt{2x+2}(\sqrt{x}+1)]^2 \times \begin{aligned} & \times \left[(x+1)^2 \left(\begin{aligned} & x^4+3x^3+8x^{5/2}+ \\ & +8x^2+8x^{3/2}+3x+1 \end{aligned} \right) \right]^2 - \\ & - \left[4x \left(\begin{aligned} & 4x^5+6x^{9/2}+6x^4+13x^{7/2}+ \\ & +18x^3+34x^{5/2}+18x^2+ \\ & +13x^{3/2}+6x+6\sqrt{x}+4 \end{aligned} \right) \right]^2. \end{aligned}$$

After simplification, we have

$$h_{28}(x) = 2(\sqrt{x}-1)^4 \times \begin{aligned} & \left(\begin{aligned} & 1+8\sqrt{x}+45x+1760x^{5/2}+698x^2+ \\ & +3616x^3+10243x^4+18541x^5+ \\ & +18541x^6+3616x^7+10243x^8+ \\ & +x^{11}+698x^9+45x^{10}+8x^{11/2}+ \\ & +6416x^{7/2}+14712x^{9/2}+208x^{3/2}+ \\ & +14712x^{13/2}+1760x^{17/2}+ \\ & +6416x^{15/2}+208x^{19/2}+20112x^{11/2} \end{aligned} \right). \end{aligned}$$

Since $h_{28}(x) > 0$, proving that $k_{28}(x) > 0$. Also we have

$$\begin{aligned} \beta &= \sup_{x \in (0, \infty)} g_{K_0M_3-\Psi M_1}(x) = \\ &= \lim_{x \rightarrow 1} g_{K_0M_3-\Psi M_1}(x) = \frac{3}{13}. \end{aligned}$$

36. For $D_{\Psi I}^{44}(P||Q) \leq \frac{40}{39}D_{\Psi M_1}^{43}(P||Q)$: Let us consider $g_{\Psi I-\Psi M_1}(x) = f''_{\Psi I}(x)/f''_{\Psi M_1}(x)$, then we have

$$g_{\Psi I-\Psi M_1}(x) = \frac{(x^2+3x+1)(x-1)^2\sqrt{2x+2}}{\left(\begin{aligned} & \sqrt{2x+2}(x^3-4x^{3/2}+1) \times \\ & \times (x+1)+4x^{3/2}(x^{3/2}+1) \end{aligned} \right)}$$

and

$$g'_{\Psi I-\Psi M_1}(x) = -\frac{2x^{3/2}\sqrt{2x+2}(x-1) \times k_{29}(x)}{(x+1) \left(\begin{aligned} & \sqrt{2x+2}(x^3-4x^{3/2}+1) \times \\ & \times (x+1)+4x^{3/2}(x^{3/2}+1) \end{aligned} \right)^2},$$

where

$$k_{29}(x) = \sqrt{2x+2}(x+1) \times \begin{aligned} & \times \left(\begin{aligned} & 3(\sqrt{x}-1)^2(x^2+1)(x+1)+ \\ & +2x^{7/2}+3x^3+10x^2+3x+2\sqrt{x} \end{aligned} \right) - \\ & - \left(\begin{aligned} & 3x^{11/2}+6x^{9/2}+6x^4+10x^{7/2}+15x^3+ \\ & +15x^{5/2}+10x^2+6x^{3/2}+6x+3 \end{aligned} \right). \end{aligned}$$

This gives

$$g'_{\Psi I-\Psi M_1}(x) \begin{cases} > 0 & x < 1 \\ < 0 & x > 1 \end{cases},$$

provides $k_{29}(x) > 0$. In order to prove $k_{29}(x) > 0$, let us consider

$$h_{29}(x) = [\sqrt{2x+2}(x+1)]^2 \times \\ \times \left(\frac{3(\sqrt{x}-1)^2(x^2+1)(x+1) + 2x^{7/2} + 3x^3 + 10x^2 + 3x + 2\sqrt{x}}{+15x^{5/2} + 10x^2 + 6x^{3/2} + 6x + 3} \right)^2 - \\ - \left(\frac{3x^{11/2} + 6x^{9/2} + 6x^4 + 10x^{7/2} + 15x^3 + 15x^{5/2} + 10x^2 + 6x^{3/2} + 6x + 3}{+15x^{5/2} + 10x^2 + 6x^{3/2} + 6x + 3} \right)^2.$$

After simplifications, we have

$$h_{29}(x) = (\sqrt{x}-1)^4 \times s(x),$$

where

$$s(x) = (x+1) \left(\frac{9 + 56x + 179x^2 + 360x^3 + 491x^4 + 491x^5 + 360x^6 + 179x^7 + 56x^8 + 9x^9}{+179x^7 + 56x^8 + 9x^9} \right) - \\ - 2\sqrt{x} \left(\frac{6 + 32x + 81x^2 + 144x^3 + 179x^4 + 144x^5 + 81x^6 + 32x^7 + 6x^8}{+144x^5 + 81x^6 + 32x^7 + 6x^8} \right).$$

We know that $x+1 \geq 2\sqrt{x}$, this allows us to conclude that

$$s(x) \geq 2\sqrt{x} \left[\left(\frac{9 + 56x + 179x^2 + 360x^3 + 491x^4 + 491x^5 + 360x^6 + 179x^7 + 56x^8 + 9x^9}{+179x^7 + 56x^8 + 9x^9} \right) - \right. \\ \left. - \left(\frac{6 + 32x + 81x^2 + 144x^3 + 179x^4 + 144x^5 + 81x^6 + 32x^7 + 6x^8}{+144x^5 + 81x^6 + 32x^7 + 6x^8} \right) \right].$$

After simplifications, we have

$$s(x) \geq 2\sqrt{x} \left(\frac{3x^8 + 15x^7 + 51x^6 + 84x^5 + 84x^4 + 84x^3 + 51x^2 + 15x + 3}{+51x^2 + 15x + 3} \right) \geq 0.$$

This gives us $h_{29}(x) > 0$. Hence $k_{29}(x) > 0$. Also we have

$$\beta = \sup_{x \in (0, \infty)} g_{\Psi I - \Psi M_1}(x) = \lim_{x \rightarrow 1} g_{\Psi I - \Psi M_1}(x) = \frac{40}{39}.$$

37. **For $D_{\Psi M_1}^{43}(\mathbf{P}||\mathbf{Q}) \leq \frac{39}{37} D_{\Psi M_2}^{42}(\mathbf{P}||\mathbf{Q})$:** Let us consider $g_{\Psi M_1 - \Psi M_2}(x) = f_{\Psi M_1}''(x)/f_{\Psi M_2}''(x)$, then we have

$$g_{\Psi M_1 - \Psi M_2}(x) = \\ = \frac{3 \left(\sqrt{2x+2} (x^3 - 4x^{3/2} + 1) \times \right. \\ \left. \times (x+1) + 4x^{3/2} (x^{3/2} + 1) \right)}{\left(\sqrt{2x+2} (3x^3 - 8x^{3/2} + 3) \times \right. \\ \left. \times (x+1) + 4x^2 (x^{3/2} + 1) \right)}$$

and

$$g'_{\Psi M_1 - \Psi M_2}(x) = \\ = - \frac{36\sqrt{x} (x^{3/2} - 1) (x+1) \times k_{30}(x)}{\left(\sqrt{2x+2} [\sqrt{2x+2} (x+1) \times \right. \\ \left. \times (3x^3 - 8x^{3/2} + 3) + 4x^{3/2} (x^{3/2} + 1)]^2 \right)},$$

where

$$k_{30}(x) = 2 \left(x^4 + 3x^{5/2} + 3x^{3/2} + 1 \right)^2 - \\ - \sqrt{2x+2} \left(x^{3/2} + 1 \right) (x+1)^2.$$

This gives

$$g'_{\Psi M_1 - \Psi M_2}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases},$$

provided $k_{30}(x) > 0$. In order to prove $k_{30}(x) > 0$, let us consider

$$h_{30}(x) = \left[2 \left(x^4 + 3x^{5/2} + 3x^{3/2} + 1 \right) \right]^2 - \\ - \left[\sqrt{2x+2} \left(x^{3/2} + 1 \right) (x+1)^2 \right]^2.$$

After simplifications, we have

$$h_{30}(x) = 2 (\sqrt{x}-1)^4 \times \\ \times \left(\frac{x^6 + 4x^{11/2} + 5x^5 + 10x^{9/2} + 15x^4 + 18x^{7/2} + 24x^3 + 18x^{5/2} + 15x^2 + 10x^{3/2} + 5x + 4\sqrt{x} + 1}{+10x^{3/2} + 5x + 4\sqrt{x} + 1} \right).$$

Since $h_{30}(x) > 0$, this gives that $k_{30}(x) > 0$. Also we have

$$\beta = \sup_{x \in (0, \infty)} g_{\Psi M_1 - \Psi M_2}(x) = \lim_{x \rightarrow 1} g_{\Psi M_1 - \Psi M_2}(x) = \frac{39}{37}.$$

38. **For $D_{\Psi M_2}^{42}(P||Q) \leq \frac{37}{36} D_{\Psi h}^{41}(P||Q)$:** Let us consider $g_{\Psi M_2 - \Psi h}(x) = f_{\Psi M_2}''(x)/f_{\Psi h}''(x)$, then we have

$$g_{\Psi M_2 - \Psi h}(x) = \\ = \frac{\left(\sqrt{2x+2} (3x^3 - 8x^{3/2} + 3) \times \right. \\ \left. \times (x+1) + 4x^{3/2} (x^{3/2} + 1) \right)}{3 \left[\sqrt{x}(x+1) \sqrt{2x+2} (x^{3/2} - 1)^2 \right]}$$

and

$$g'_{\Psi M_2 - \Psi h}(x) = - \frac{\sqrt{x} \times k_{31}(x)}{\sqrt{2x+2} (x+1)^2 (x^{3/2} - 1)^3},$$

where $k_{31}(x) = k_{30}(x) > 0$. This gives

$$g'_{\Psi M_2 - \Psi h}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases}.$$

Also we have

$$\beta = \sup_{x \in (0, \infty)} g_{\Psi M_2 - \Psi h}(x) = \lim_{x \rightarrow 1} g_{\Psi M_2 - \Psi h}(x) = \frac{37}{36}.$$

39. For $D_{\Psi h}^{41}(\mathbf{P}||\mathbf{Q}) \leq \frac{9}{8}D_{\Psi J}^{39}(\mathbf{P}||\mathbf{Q})$: It is true in view of (2).

40. For $D_{\Psi h}^{41}(\mathbf{P}||\mathbf{Q}) \leq \frac{12}{11}D_{\Psi M_3}^{40}(\mathbf{P}||\mathbf{Q})$: Let us consider $g_{\Psi h-\Psi M_3}(x) = f''_{\Psi h}(x)/f''_{\Psi M_3}(x)$, then we have

$$\begin{aligned} g_{\Psi h-\Psi M_3}(x) &= \frac{(x^{3/2} - 1)^2 (x+1) \sqrt{2x+2}}{\left(\frac{\sqrt{2x+2} (x^3+1) \times}{\times (x+1) - 4x^{3/2} (x^{3/2}+1)} \right)} \end{aligned}$$

and

$$\begin{aligned} g'_{\Psi h-\Psi M_3}(x) &= - \frac{3x^{3/2} \sqrt{2x+2} (x^{3/2} - 1) \times k_{32}(x)}{\left(\frac{\sqrt{2x+2} (x^3+1) \times}{\times (x+1) - 4x^{3/2} (x^{3/2}+1)} \right)^2}, \end{aligned}$$

where $k_{32}(x) = k_{30}(x) > 0$. This gives

$$g'_{\Psi h-\Psi M_3}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases}.$$

Also we have

$$\beta = \sup_{x \in (0, \infty)} g_{\Psi h-\Psi M_3}(x) = \lim_{x \rightarrow 1} g_{\Psi h-\Psi M_3}(x) = \frac{12}{11}.$$

41. For $D_{\Psi M_3}^{40}(\mathbf{P}||\mathbf{Q}) \leq \frac{11}{8}D_{\Psi K_0}^{37}(\mathbf{P}||\mathbf{Q})$: Let us consider $g_{\Psi M_3-\Psi K_0}(x) = f''_{\Psi M_3}(x)/f''_{\Psi K_0}(x)$, then we have

$$\begin{aligned} g_{\Psi M_3-\Psi K_0}(x) &= \frac{4 \left(\frac{\sqrt{2x+2} (x^3+1) \times}{\times (x+1) - 4x^{3/2} (x^{3/2}+1)} \right)}{\left(\frac{(x+1) \sqrt{2x+2} (\sqrt{x}-1)^2 \times}{\times (4x^2+5x^{3/2}+6x+5\sqrt{x}+4)} \right)} \end{aligned}$$

and

$$\begin{aligned} g'_{\Psi M_3-\Psi K_0}(x) &= - \frac{6 \times k_{33}(x)}{\left(\frac{(\sqrt{x}-1)^3 \sqrt{2x+2} \sqrt{x} (x+1)^2 \times}{\times (4x^2+5x^{3/2}+6x+5\sqrt{x}+4)} \right)^2}, \end{aligned}$$

where

$$\begin{aligned} k_{33}(x) &= \left(\frac{\sqrt{2x+2} (\sqrt{x}+1) (x+1)^2 \times}{\times (1+3x+8x^2+3x^3+x^4)} \right) - \\ &- 4x \left(\frac{4+2x^4+2x+6x^2+11x^{3/2}+2\sqrt{x}+}{+14x^{5/2}+6x^3+11x^{7/2}+4x^5+2x^{9/2}} \right). \end{aligned}$$

This gives

$$g'_{\Psi M_3-\Psi K_0}(x) \begin{cases} > 0 & x < 1 \\ < 0 & x > 1 \end{cases},$$

provided $k_{33}(x) > 0$. In order to prove $k_{33}(x) > 0$, let us consider

$$\begin{aligned} h_{33}(x) &= \left[\frac{\sqrt{2x+2} (\sqrt{x}+1) (x+1)^2 \times}{\times (1+3x+8x^2+3x^3+x^4)} \right]^2 - \\ &- \left[4x \left(\frac{4+2x^4+2x+6x^2+11x^{3/2}+}{+2\sqrt{x}+14x^{5/2}+6x^3+}{+11x^{7/2}+4x^5+2x^{9/2}} \right) \right]^2. \end{aligned}$$

After simplifications, we have

$$\begin{aligned} h_{33}(x) &= (\sqrt{x}-1)^4 \times \\ &\times \left(\begin{aligned} &1+30x+2144x^5+6\sqrt{x}+231x^2+ \\ &+698x^3+380x^{5/2}+110x^{3/2}+ \\ &+1056x^{7/2}+1475x^4+1874x^{9/2}+ \\ &+2298x^6+698x^9+30x^{11}+231x^{10}+ \\ &+2144x^7+380x^{19/2}+1056x^{17/2}+ \\ &+1475x^8+x^{12}+6x^{23/2}+110x^{21/2}+ \\ &+2366x^{13/2}+2366x^{11/2}+1874x^{15/2} \end{aligned} \right). \end{aligned}$$

Since $h_{33}(x) > 0$, this gives that $k_{33}(x) > 0$. Also we have

$$\beta = \sup_{x \in (0, \infty)} g_{\Psi M_3-\Psi K_0}(x) = \lim_{x \rightarrow 1} g_{\Psi M_3-\Psi K_0}(x) = \frac{11}{8}.$$

42. For $D_{\Psi J}^{39}(\mathbf{P}||\mathbf{Q}) \leq \frac{4}{3}D_{\Psi K_0}^{37}(\mathbf{P}||\mathbf{Q})$: Let us consider $g_{\Psi J-\Psi K_0}(x) = f''_{\Psi J}(x)/f''_{\Psi K_0}(x)$, then we have

$$g_{\Psi J-\Psi K_0}(x) = \frac{4(\sqrt{x}+1)^2 (x+1)}{4x^2+5x^{3/2}+6x+5\sqrt{x}+4},$$

$$\begin{aligned} g'_{\Psi J-\Psi K_0}(x) &= - \frac{2(x-1) \left(\frac{3x^2+4x^{3/2}+}{+10x+4\sqrt{x}+3} \right)}{\sqrt{x} \left(\frac{4x^2+5x^{3/2}+}{+6x+5\sqrt{x}+4} \right)^2} \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases} \end{aligned}$$

and

$$\beta = \sup_{x \in (0, \infty)} g_{\Psi J-\Psi K_0}(x) = \lim_{x \rightarrow 1} g_{\Psi J-\Psi K_0}(x) = \frac{4}{3}.$$

43. For $D_{\Psi K_0}^{37}(\mathbf{P}||\mathbf{Q}) \leq D_{\Psi T}^{38}(\mathbf{P}||\mathbf{Q})$: It is true in view of pyramid.

44. For $D_{\Psi K_0}^{37}(\mathbf{P}||\mathbf{Q}) \leq \frac{1}{3}D_{\Psi F\Delta}^{55}(\mathbf{P}||\mathbf{Q})$: Let us consider $g_{\Psi K_0-F\Delta}(x) = f''_{\Psi K_0}(x)/f''_{F\Delta}(x)$, then we have

$$\begin{aligned} g_{\Psi K_0-F\Delta}(x) &= \frac{4\sqrt{x} \left(\frac{4x^2+5x^{3/2}+}{+6x+5\sqrt{x}+4} \right) (x+1)^3}{\left(\begin{aligned} &15+90x+257x^2+492x^3+ \\ &+257x^4+90x^5+15x^6+30\sqrt{x}+ \\ &+150x^{3/2}+364x^{5/2}+ \\ &+364x^{7/2}+150x^{9/2}+30x^{11/2} \end{aligned} \right)}, \end{aligned}$$

$$\begin{aligned}
g'_{\Psi K_0-F\Delta}(x) &= \\
&= -\frac{12(x+1)^2(x-1)^3(\sqrt{x}-1)^2}{\sqrt{x} \left(\begin{aligned} &15+90x+257x^2+492x^3+ \\ &+257x^4+90x^5+15x^6+ \\ &+30\sqrt{x}+150x^{3/2}+364x^{5/2}+ \\ &+364x^{7/2}+150x^{9/2}+30x^{11/2} \end{aligned} \right)^2} \times \\
&\times \left(\begin{aligned} &10+110x+486x^2+ \\ &+740x^3+486x^4+ \\ &+110x^5+10x^6+25\sqrt{x}+ \\ &+205x^{3/2}+586x^{5/2}+ \\ &+586x^{7/2}+205x^{9/2}+25x^{11/2} \end{aligned} \right) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases}
\end{aligned}$$

and

$$\beta = \sup_{x \in (0, \infty)} g_{\Psi K_0-F\Delta}(x) = \lim_{x \rightarrow 1} g_{\Psi K_0-F\Delta}(x) = \frac{1}{3}.$$

45. For $\mathbf{D}_{\Psi T}^{38}(\mathbf{P}||\mathbf{Q}) \leq \frac{3}{8}\mathbf{D}_{FI}^{54}(\mathbf{P}||\mathbf{Q})$: Let us consider $g_{\Psi T-FI}(x) = f''_{\Psi T}(x)/f''_{FI}(x)$, then we have

$$\begin{aligned}
g_{\Psi T-FI}(x) &= \\
&= \frac{16\sqrt{x}(x^2+x+1)(\sqrt{x}+1)^2}{\left(\begin{aligned} &15x^4+30x^{7/2}+60x^3+90x^{5/2}+ \\ &+122x^2+90x^{3/2}+60x+30\sqrt{x}+15 \end{aligned} \right)}
\end{aligned}$$

and

$$\begin{aligned}
g'_{\Psi T-FI}(x) &= \\
&= -\frac{8(x-1)}{\sqrt{x} \left(\begin{aligned} &15x^4+30x^{7/2}+60x^3+ \\ &+90x^{5/2}+122x^2+90x^{3/2}+ \\ &+60x+30\sqrt{x}+15 \end{aligned} \right)^2} \times \\
&\times \left(\begin{aligned} &15x^6+26x^{11/2}+40x^5+86x^{9/2}+9x^4+ \\ &+9x^2+86x^{3/2}+40x+26\sqrt{x}+15+ \\ &+(x^2-1)^2\sqrt{x}(34x+65\sqrt{x}+34) \end{aligned} \right).
\end{aligned}$$

This gives

$$g_{\Psi T-FI}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases}.$$

Also, we have

$$\beta = \sup_{x \in (0, \infty)} g_{\Psi T-FI}(x) = \lim_{x \rightarrow 1} g_{\Psi T-FI}(x) = \frac{3}{8}.$$

46. For $\mathbf{D}_{F\Delta}^{55}(\mathbf{P}||\mathbf{Q}) \leq \frac{9}{8}\mathbf{D}_{FI}^{54}(\mathbf{P}||\mathbf{Q})$: Let us consider $g_{\Psi \Delta-FI}(x) = f''_{\Psi \Delta}(x)/f''_{FI}(x)$, then we have

$$\begin{aligned}
g_{F\Delta-FI}(x) &= \\
&= \frac{\left(\begin{aligned} &15+90x+257x^2+492x^3+ \\ &+257x^4+90x^5+15x^6+ \\ &+30\sqrt{x}+150x^{3/2}+364x^{5/2}+ \\ &+364x^{7/2}+150x^{9/2}+30x^{11/2} \end{aligned} \right)}{(x+1)^2 \left(\begin{aligned} &15x^4+30x^{7/2}+60x^3+ \\ &+90x^{5/2}+122x^2+90x^{3/2}+ \\ &+60x+30\sqrt{x}+15 \end{aligned} \right)}
\end{aligned}$$

$$\begin{aligned}
g'_{F\Delta-FI}(x) &= \\
&= -\frac{32x^{3/2}(x-1)}{(x+1)^3 \left(\begin{aligned} &15x^4+30x^{7/2}+60x^3+ \\ &+90x^{5/2}+122x^2+90x^{3/2}+ \\ &+60x+30\sqrt{x}+15 \end{aligned} \right)^2} \times \\
&\times \left(\begin{aligned} &75x^5+300x^{9/2}+675x^4+1200x^{7/2}+ \\ &+1682x^3+1928x^{5/2}+1682x^2+ \\ &+1200x^{3/2}+675x+300\sqrt{x}+75 \end{aligned} \right).
\end{aligned}$$

This gives

$$g'_{F\Delta-FI}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases}.$$

Also, we have

$$\beta = \sup_{x \in (0, \infty)} g_{F\Delta-FI}(x) = \lim_{x \rightarrow 1} g_{F\Delta-FI}(x) = \frac{9}{8}.$$

47. For $\mathbf{D}_{FI}^{54}(\mathbf{P}||\mathbf{Q}) \leq \frac{64}{83}\mathbf{D}_{FM_1}^{53}(\mathbf{P}||\mathbf{Q})$: Let us consider $g_{FI-FM_1}(x) = f''_{FI}(x)/f''_{FM_1}(x)$, then we have

$$\begin{aligned}
g_{FI-FM_1}(x) &= \\
&= \frac{(\sqrt{x}-1)^2\sqrt{2x+2}}{\left(\begin{aligned} &\sqrt{2x+2}(15x^4-62x^2+15) \times \\ &\times (x+1)+64x^2(x^{3/2}+1) \end{aligned} \right)} \times \\
&\times \left(\begin{aligned} &15x^4+30x^{7/2}+60x^3+90x^{5/2}+ \\ &+122x^2+90x^{3/2}+60x+30\sqrt{x}+15 \end{aligned} \right)
\end{aligned}$$

and

$$\begin{aligned}
g'_{FI-FM_1}(x) &= \\
&= -\frac{64(\sqrt{x}-1) \times k_{34}(x)}{\left(\begin{aligned} &\sqrt{2x+2}(15x^4-62x^2+15) \times \\ &\times (x+1)+64x^2(x^{3/2}+1) \end{aligned} \right)^2},
\end{aligned}$$

where

$$\begin{aligned}
k_{34}(x) &= \sqrt{2x+2} \times \\
&\times \left(\begin{aligned} &-15x^6+242x^{5/2}+165x+ \\ &+302x^3+45x^5+242x^4+ \\ &+225x^2+225x^{9/2}+60+ \\ &+45x^{3/2}+60x^{13/2}-15\sqrt{x}+ \\ &+302x^{7/2}+165x^{11/2} \end{aligned} \right) - \\
&- \left(\begin{aligned} &60+135x+255x^5+60\sqrt{x}+255x^2+ \\ &+478x^3+484x^{5/2}+240x^{3/2}+ \\ &+672x^{7/2}+478x^4+484x^{9/2}+ \\ &+135x^6+60x^{13/2}+240x^{11/2}+60x^7 \end{aligned} \right).
\end{aligned}$$

This gives

$$g'_{FI-FM_1}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases},$$

provided $k_{34}(x) > 0$. In order to prove $k_{34}(x) > 0$, let us consider

$$h_{34}(x) = (\sqrt{2x+2})^2 \times \left(\begin{aligned} &-15x^6 + 242x^{5/2} + 165x + 302x^3 + \\ &+45x^5 + 242x^4 + 225x^2 + 225x^{9/2} + \\ &+60 + 45x^{3/2} + 60x^{13/2} - 15\sqrt{x} + \\ &+302x^{7/2} + 165x^{11/2} \end{aligned} \right)^2 - \left(\begin{aligned} &60 + 135x + 255x^5 + 60\sqrt{x} + 255x^2 + \\ &+478x^3 + 484x^{5/2} + 240x^{3/2} + \\ &+672x^{7/2} + 478x^4 + 484x^{9/2} + 135x^6 + \\ &+60x^{13/2} + 240x^{11/2} + 60x^7 \end{aligned} \right)^2.$$

After simplification, we get

$$h_{34}(x) = (\sqrt{x}-1)^4 \times \left(\begin{aligned} &3600 + 441250x^6 + 20250x + \\ &+396832x^5 + 3600\sqrt{x} + 61875x^2 + \\ &+147270x^3 + 90000x^{5/2} + 26100x^{3/2} + \\ &+208260x^{7/2} + 277740x^4 + \\ &+352860x^{9/2} + 20250x^{11} + \\ &+451980x^{11/2} + 451980x^{13/2} + \\ &+352860x^{15/2} + 3600x^{23/2} + 3600x^{12} + \\ &+147270x^9 + 396832x^7 + \\ &+26100x^{21/2} + 277740x^8 + \\ &+208260x^{17/2} + 90000x^{19/2} + 61875x^{10} \end{aligned} \right).$$

Since $h_{34}(x) > 0$, this gives that $k_{34}(x) > 0$. Also we have

$$\beta = \sup_{x \in (0, \infty)} g_{FI-FM_1}(x) = \lim_{x \rightarrow 1} g_{FI-FM_1}(x) = \frac{64}{63}.$$

48. For $\mathbf{D}_{\mathbf{FM}_1}^{53}(\mathbf{P}||\mathbf{Q}) \leq \frac{63}{61}\mathbf{D}_{\mathbf{FM}_2}^{52}(\mathbf{P}||\mathbf{Q})$: Let us consider $g_{FM_1-FM_2}(x) = f''_{FM_1}(x)/f''_{FM_2}(x)$, then we have

$$g_{FM_1-FM_2}(x) = \frac{3 \left(\begin{aligned} &\sqrt{2x+2} (15x^4 - 62x^2 + 15) \times \\ &\times (x+1) + 64x^2 (x^{3/2} + 1) \end{aligned} \right)}{\left(\begin{aligned} &\sqrt{2x+2} (45x^4 - 122x^2 + 45) \times \\ &\times (x+1) + 64x^2 (x^{3/2} + 1) \end{aligned} \right)}$$

and

$$g'_{FM_1-FM_2}(x) = -\frac{5760x(x-1)(x+1)^3}{\sqrt{2x+2}} \times \frac{k_{35}(x)}{\left(\begin{aligned} &\sqrt{2x+2} (45x^4 - 122x^2 + 45) \times \\ &\times (x+1) + 64x^2 (x^{3/2} + 1) \end{aligned} \right)^2},$$

where

$$k_{35}(x) = (\sqrt{x}+1) \times \left[x^2 (2\sqrt{x}-1)^2 + 6x^{3/2} + (\sqrt{x}-2)^2 \right] - 2\sqrt{2x+2} (x^2+1)(x+1).$$

This gives

$$g'_{FM_1-FM_2}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases},$$

provided $k_{35}(x) > 0$. In order to prove $k_{35}(x) > 0$, let us consider

$$h_{35}(x) = (\sqrt{x}+1)^2 \times \left[x^2 (2\sqrt{x}-1)^2 + 6x^{3/2} + (\sqrt{x}-2)^2 \right]^2 - [2\sqrt{2x+2} (x^2+1)(x+1)]^2.$$

After simplifications, we have

$$h_{35}(x) = (\sqrt{x}-1)^4 \times \left(\begin{aligned} &8x^5 + 32x^{9/2} + 32x^4 + 24x^{7/2} + \\ &+49x^3 + 82x^{5/2} + 49x^2 + \\ &+24x^{3/2} + 32x + 32\sqrt{x} + 8 \end{aligned} \right).$$

Since $h_{35}(x) > 0$, this gives that $k_{35}(x) > 0$. Also we have

$$\beta = \sup_{x \in (0, \infty)} g_{FM_1-FM_2}(x) = \lim_{x \rightarrow 1} g_{FM_1-FM_2}(x) = \frac{63}{61}.$$

49. For $\mathbf{D}_{\mathbf{FM}_2}^{52}(\mathbf{P}||\mathbf{Q}) \leq \frac{61}{60}\mathbf{D}_{\mathbf{Fh}}^{51}(\mathbf{P}||\mathbf{Q})$: Let us consider $g_{FM_2-Fh}(x) = f''_{FM_2}(x)/f''_{Fh}(x)$, then we consider

$$g_{FM_2-Fh}(x) = \frac{\left(\begin{aligned} &\sqrt{2x+2} (45x^4 - 122x^2 + 45) \times \\ &\times (x+1) + 64x^2 (x^{3/2} + 1) \end{aligned} \right)}{45\sqrt{2x+2} (x+1)^3 (x-1)^2}$$

and

$$g'_{FM_2-Fh}(x) = -\frac{32x \times k_{36}(x)}{45(x-1)^3 \sqrt{2x+2} (x+1)^4}$$

where $k_{36}(x) = k_{35}(x) > 0$. This gives

$$g'_{FM_2-Fh}(x) \begin{cases} > 0 & x < 1 \\ < 0 & x > 1 \end{cases}.$$

Also we have

$$\beta = \sup_{x \in (0, \infty)} g_{FM_2-Fh}(x) = \lim_{x \rightarrow 1} g_{FM_2-Fh}(x) = \frac{61}{60}.$$

50. For $\mathbf{D}_{\mathbf{Fh}}^{51}(\mathbf{P}||\mathbf{Q}) \leq \frac{15}{14}\mathbf{D}_{\mathbf{FJ}}^{49}(\mathbf{P}||\mathbf{Q})$: Let us consider $g_{Fh-FJ}(x) = f''_{Fh}(x)/f''_{FJ}(x)$, then we have

$$g_{Fh-FJ}(x) = \frac{15(\sqrt{x}+1)^2 (x+1)^2}{\left(\begin{aligned} &15x^3 + 30x^{5/2} + 45x^2 + \\ &+44x^{3/2} + 45x + 30\sqrt{x} + 15 \end{aligned} \right)},$$

and

$$g'_{Fh-FJ}(x) = -\frac{120(x-1)(x+1)\sqrt{x}(3x+4\sqrt{x}+3)}{\left(\begin{matrix} 15x^3+30x^{5/2}+45x^2+ \\ +44x^{3/2}+45x+30\sqrt{x}+15 \end{matrix}\right)^2}.$$

This gives

$$g'_{Fh-FJ}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases}.$$

Also, we have

$$\beta = \sup_{x \in (0, \infty)} g_{Fh-FJ}(x) = \lim_{x \rightarrow 1} g_{Fh-FJ}(x) = \frac{15}{14}.$$

51. For $D_{Fh}^{51}(\mathbf{P}||\mathbf{Q}) \leq \frac{20}{19} D_{FM_3}^{50}(\mathbf{P}||\mathbf{Q})$: Let us consider $g_{Fh-FM_3}(x) = f''_{Fh}(x)/f''_{FM_3}(x)$, then we have

$$g_{Fh-FM_3}(x) = \frac{15(x-1)^2(x+1)^3\sqrt{2x+2}}{\left(\begin{matrix} \sqrt{2x+2}(15x^4+2x^2+15) \times \\ \times (x+1) - 64x^2(x^{3/2}+1) \end{matrix}\right)^2}$$

and

$$g'_{Fh-FM_3}(x) = -\frac{960x(x-1)(x+1)^2 \times k_{37}(x)}{\left(\begin{matrix} \sqrt{2x+2}[-64x^2(x^{3/2}+1) + \sqrt{2x+2} \times \\ \times (x+1)(15x^4+2x^2+15)]^2 \end{matrix}\right)},$$

Where $k_{37}(x) = k_{35}(x) > 0$. This gives

$$g'_{Fh-FM_3}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases}.$$

Also we have

$$\beta = \sup_{x \in (0, \infty)} g_{Fh-FM_3}(x) = \lim_{x \rightarrow 1} g_{Fh-FM_3}(x) = \frac{20}{19}.$$

52. For $D_{FM_3}^{50}(\mathbf{P}||\mathbf{Q}) \leq \frac{19}{16} D_{FK_0}^{47}(\mathbf{P}||\mathbf{Q})$: Let us consider $g_{FM_3-FK_0}(x) = f''_{FM_3}(x)/f''_{FK_0}(x)$, then we have

$$g_{FM_3-FK_0}(x) = \frac{\left(\begin{matrix} \sqrt{2x+2}(15x^4+2x^2+15) \times \\ \times (x+1) - 64x^2(x^{3/2}+1) \end{matrix}\right)}{3(x+1)(x-1)^2\sqrt{2x+2}(5x^2+6x+5)}$$

and

$$g'_{FM_3-FK_0}(x) = -\frac{4 \times k_{38}(x)}{\left(\begin{matrix} 3(x-1)^3\sqrt{x}\sqrt{2x+2} \times \\ \times (x+1)^2(5x^2+6x+5)^2 \end{matrix}\right)},$$

where

$$k_{38}(x) = \sqrt{2x+2}(x+1)^3 \times (15x^4+20x^3+58x^2+20x+15) - 8x(\sqrt{x}+1) \left(\begin{matrix} 10x^5+(\sqrt{x}-1)^2 \times \\ (10x^4+x^3+x+10) + \\ +26x^4+22x^3+12x^{5/2}+ \\ +22x^2+26x+10 \end{matrix}\right).$$

This gives

$$g'_{FM_3-FK_0}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases},$$

provided $k_{38}(x) > 0$. In order to prove $k_{38}(x) > 0$, let us consider

$$h_{38}(x) = [\sqrt{2x+2}(x+1)^3]^2 \times (15x^4+20x^3+58x^2+20x+15)^2 - \left[8x(\sqrt{x}+1) \left(\begin{matrix} 10x^5+(\sqrt{x}-1)^2 \times \\ \times (10x^4+x^3+x+10) + \\ +26x^4+22x^3+12x^{5/2}+ \\ +22x^2+26x+10 \end{matrix}\right)\right]^2.$$

After simplifications, we have

$$h_{38}(x) = 2(\sqrt{x}-1)^4 \times \left(\begin{matrix} 225+4425x+27890x^2+135760x^{7/2}+ \\ +349215x^5+900\sqrt{x}+94290x^3+ \\ +27890x^{11}+436976x^{13/2}+389920x^{15/2}+ \\ +349215x^8+208791x^9+13200x^{3/2}+ \\ +49160x^{5/2}+225x^{13}+265724x^{17/2}+ \\ +208791x^4+453852x^7+389920x^{11/2}+ \\ +4425x^{12}+94290x^{10}+3200x^{23/2}+ \\ +900x^{25/2}+49160x^{21/2}+ \\ +135760x^{19/2}+453852x^6+265724x^{9/2} \end{matrix}\right).$$

Since $h_{38}(x) > 0$, proving that $k_{38}(x) > 0$. Also we have

$$\beta = \sup_{x \in (0, \infty)} g_{FM_3-FK_0}(x) = \lim_{x \rightarrow 1} g_{FM_3-FK_0}(x) = \frac{19}{16}.$$

53. For $D_{FJ}^{49}(\mathbf{P}||\mathbf{Q}) \leq \frac{7}{6} D_{FK_0}^{47}(\mathbf{P}||\mathbf{Q})$: Let us consider $g_{FJ-FK_0}(x) = f''_{FJ}(x)/f''_{FK_0}(x)$, then we have

$$g_{FJ-FK_0}(x) = \frac{\left(\begin{matrix} 15x^3+30x^{5/2}+45x^2+ \\ +44x^{3/2}+45x+30\sqrt{x}+15 \end{matrix}\right)}{3(5x^2+6x+5)(\sqrt{x}+1)^2}$$

and

$$g'_{FJ-FK_0}(x) = -\frac{4(\sqrt{x}-1) \left(\begin{matrix} 15x^3+30x^{5/2}+65x^2+ \\ +68x^{3/2}+65x+30\sqrt{x}+15 \end{matrix}\right)}{3(5x^2+6x+5)^2(\sqrt{x}+1)^3}.$$

This gives

$$g'_{FJ\text{-}FK_0}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases}.$$

Also, we have

$$\beta = \sup_{x \in (0, \infty)} g_{FJ\text{-}FK_0}(x) = \lim_{x \rightarrow 1} g_{FJ\text{-}FK_0}(x) = \frac{7}{6}.$$

54. **For $D_{FK_0}^{47}(\mathbf{P}||\mathbf{Q}) \leq D_{FT}^{48}(\mathbf{P}||\mathbf{Q})$:** It holds in view of pyramid.

55. **For $D_{FT}^{48}(\mathbf{P}||\mathbf{Q}) \leq 2D_{F\Psi}^{46}(\mathbf{P}||\mathbf{Q})$:** Let us consider $g_{FK_0\text{-}FT}(x) = f''_{FK_0}(x)/f''_{FT}(x)$, then we have

$$g_{FT\text{-}F\Psi}(x) = \frac{\begin{pmatrix} 15x^4 + 30x^{7/2} + 60x^3 + 58x^{5/2} + \\ + 58x^2 + 58x^{3/2} + 60x + 30\sqrt{x} + 15 \end{pmatrix}}{(x+1) \begin{pmatrix} 15x^3 + 14x^{5/2} + 13x^2 + \\ + 12x^{3/2} + 13x + 14\sqrt{x} + 15 \end{pmatrix}}$$

and

$$g'_{FT\text{-}F\Psi}(x) = -\frac{8(x-1)}{\sqrt{x}(x+1)^2 \begin{pmatrix} 15x^3 + 14x^{5/2} + \\ + 13x^2 + 12x^{3/2} + \\ + 13x + 14\sqrt{x} + 15 \end{pmatrix}^2} \times \begin{pmatrix} 15x^6 + 60x^{11/2} + 105x^5 + 184x^{9/2} + \\ + 265x^4 + 380x^{7/2} + 382x^3 + 380x^{5/2} + \\ + 265x^2 + 184x^{3/2} + 105x + 60\sqrt{x} + 15 \end{pmatrix}.$$

This gives

$$g'_{FT\text{-}F\Psi}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases}.$$

Also, we have

$$\beta = \sup_{x \in (0, \infty)} g_{FT\text{-}F\Psi}(x) = \lim_{x \rightarrow 1} g_{FT\text{-}F\Psi}(x) = 2.$$

56. **For $D_{Jh}^{17}(\mathbf{P}||\mathbf{Q}) \leq 4D_{JM_3}^{16}(\mathbf{P}||\mathbf{Q})$:** Let us consider $g_{Jh\text{-}JM_3}(x) = f''_{Jh}(x)/f''_{JM_3}(x)$, then we have

$$g_{Jh\text{-}JM_3}(x) = \frac{(\sqrt{x}-1)^2(x+1)\sqrt{2x+2}}{\begin{pmatrix} \sqrt{2x+2}(x+1)^2 - \\ -4\sqrt{x}(x^{3/2}+1) \end{pmatrix}}$$

and

$$g'_{Jh\text{-}JM_3}(x) = \frac{2(1-\sqrt{x})(x+1) \times k_{39}(x)}{\begin{pmatrix} \sqrt{x}\sqrt{2x+2}[\sqrt{2x+2} \times \\ \times (x+1)^2 - 4\sqrt{x}(x^{3/2}+1)]^2 \end{pmatrix}},$$

where $k_{39}(x) = k_{21}(x) > 0$. This gives

$$g'_{Jh\text{-}JM_3}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases}.$$

Also, we have

$$\beta = \sup_{x \in (0, \infty)} g_{Jh\text{-}JM_3}(x) = \lim_{x \rightarrow 1} g_{Jh\text{-}JM_3}(x) = 4.$$

57. **For $D_{M_1I}^2(\mathbf{P}||\mathbf{Q}) \leq D_{JM_3}^{16}(\mathbf{P}||\mathbf{Q})$:** Let us consider $g_{M_1I\text{-}JM_3}(x) = f''_{M_1I}(x)/f''_{JM_3}(x)$, then we have

$$g_{M_1I\text{-}JM_3}(x) = \frac{4\sqrt{x}[\sqrt{2x+2}(x-\sqrt{x}+1) - (x^{3/2}+1)]}{\sqrt{2x+2}(x+1)^2 - 4\sqrt{x}(x^{3/2}+1)}$$

and

$$g'_{M_1I\text{-}JM_3}(x) = -\frac{2\sqrt{x}(\sqrt{x}-1)^3}{(x+1)} \times \frac{k_{40}(x)}{[\sqrt{2x+2}(x+1)^2 - 4\sqrt{x}(x^{3/2}+1)]^2},$$

where

$$k_{40}(x) = \sqrt{2x+2}(\sqrt{x}+1)(x+1)^2 - (x^3 + 3x^{5/2} + 8x^{3/2} + 3\sqrt{x} + 1).$$

This gives

$$g'_{M_1I\text{-}JM_3}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases},$$

provided $k_{40}(x) > 0$. In order to prove $k_{40}(x) > 0$, let us consider

$$h_{40}(x) = [\sqrt{2x+2}(\sqrt{x}+1)(x+1)^2]^2 - (x^3 + 3x^{5/2} + 8x^{3/2} + 3\sqrt{x} + 1)^2.$$

After simplifications, we have

$$h_{40}(x) = (\sqrt{x}-1)^4 \times \begin{pmatrix} x^5 + 4x^{9/2} + 8x^4 + 20x^{7/2} + \\ + 24x^3 + 34x^{5/2} + 24x^2 + \\ + 20x^{3/2} + 8x + 8\sqrt{x} + 1 \end{pmatrix}.$$

Since $h_{40}(x) > 0$, proving that $k_{40}(x) > 0$. Also we have

$$\beta = \sup_{x \in (0, \infty)} g_{M_1I\text{-}JM_3}(x) = \lim_{x \rightarrow 1} g_{M_1I\text{-}JM_3}(x) = 1.$$

58. For $D_{\text{TM}_3}^{23}(\mathbf{P}||\mathbf{Q}) \leq 9D_{\text{JM}_3}^{16}(\mathbf{P}||\mathbf{Q})$: Let us consider $g_{\text{TM}_3-\text{JM}_3}(x) = f_{\text{TM}_3}''(x)/f_{\text{JM}_3}''(x)$, then we have

$$g_{\text{TM}_3-\text{JM}_3}(x) = \frac{2 \left[(x^2 + 1) \sqrt{2x+2} - 2\sqrt{x} (x^{3/2} + 1) \right]}{\sqrt{2x+2} (x+1)^2 - 4\sqrt{x} (x^{3/2} + 1)}$$

and

$$g'_{\text{TM}_3-\text{JM}_3}(x) = -\frac{2\sqrt{2x+2}(x-1) \times k_{41}(x)}{\left((x+1) \sqrt{x} [\sqrt{2x+2} \times \times (x+1)^2 - 4\sqrt{x} (x^{3/2} + 1)]^2 \right)},$$

where

$$k_{41}(x) = \left(x^{7/2} + 3x^{5/2} + 4x^2 + 4x^{3/2} + 3x + 1 \right) - 2\sqrt{x}\sqrt{2x+2}(x+1)^2.$$

This gives

$$g'_{\text{TM}_3-\text{JM}_3}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases},$$

provided $k_{41}(x) > 0$. In order to prove $k_{41}(x) > 0$, let us consider

$$h_{41}(x) = \left(\frac{x^{7/2} + 3x^{5/2} + 4x^2 + 4x^{3/2} + 3x + 1}{+4x^{3/2} + 3x + 1} \right)^2 - [2\sqrt{x}\sqrt{2x+2}(x+1)^2]^2.$$

After simplifications, we have

$$h_{41}(x) = (\sqrt{x} - 1)^4 \times \left(\frac{x^5 + 4x^{9/2} + 8x^4 + 20x^{7/2} + 24x^3 + 34x^{5/2} + 24x^2 + 20x^{3/2} + 8x + 4\sqrt{x} + 1}{+34x^{5/2} + 24x^2 + 20x^{3/2} + 8x + 4\sqrt{x} + 1} \right).$$

Since $h_{41}(x) > 0$, proving that $k_{41}(x) > 0$. Also we have

$$\beta = \sup_{x \in (0, \infty)} g_{\text{TM}_3-\text{JM}_3}(x) = \lim_{x \rightarrow 1} g_{\text{TM}_3-\text{JM}_3}(x) = 9.$$

59. For $D_{\text{JM}_3}^{16}(\mathbf{P}||\mathbf{Q}) \leq \frac{1}{24}D_{\text{F}\Psi}^{46}(\mathbf{P}||\mathbf{Q})$: Let us consider $g_{\text{JM}_3-\text{F}\Psi}(x) = f_{\text{JM}_3}''(x)/f_{\text{F}\Psi}''(x)$, then we have

$$g_{\text{JM}_3-\text{F}\Psi}(x) = \frac{16x^{3/2} \left[(x+1)^2 \sqrt{2x+2} - 4\sqrt{x} (x^{3/2} + 1) \right]}{\left(\frac{\sqrt{2x+2}(x+1)(\sqrt{x}-1)^2 \times (15x^3 + 15)}{+14x^{5/2} + 13x^2 + 12x^{3/2} + 13x + 14\sqrt{x}} \right)}$$

and

$$g'_{\text{JM}_3-\text{F}\Psi}(x) = -\frac{8\sqrt{x}}{\sqrt{2x+2}(x+1)^2(\sqrt{x}-1)^3} \times \frac{k_{42}(x)}{\left(\frac{15x^3 + 15 + 14x^{5/2} + 13x^2 + 12x^{3/2} + 13x + 14\sqrt{x}}{+12x^{3/2} + 13x + 14\sqrt{x}} \right)},$$

where

$$k_{42}(x) = \sqrt{2x+2}(x+1)^2 \times \left(\frac{45x^{9/2} + 13x^4 + 88x^{7/2} + 24x^3 + 22x^{5/2} + 22x^2 + 24x^{3/2} + 88x + 13\sqrt{x} + 45}{+22x^2 + 24x^{3/2} + 88x + 13\sqrt{x} + 45} \right) - 12\sqrt{x} \left(\frac{20x^6 + 4x^{11/2} + 9x^5 + 44x^{9/2} + 12x^4 + 32x^{7/2} + 14x^3 + 32x^{5/2} + 12x^2 + 44x^{3/2} + 9x + 4\sqrt{x} + 20}{+12x^2 + 44x^{3/2} + 9x + 4\sqrt{x} + 20} \right).$$

This gives

$$g'_{\text{JM}_3-\text{F}\Psi}(x) \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases},$$

provided $k_{42}(x) > 0$. In order to prove $k_{42}(x) > 0$, let us consider

$$h_{42}(x) = [\sqrt{2x+2}(x+1)^2]^2 \times \left(\frac{45x^{9/2} + 13x^4 + 88x^{7/2} + 24x^3 + 22x^{5/2} + 22x^2 + 24x^{3/2} + 88x + 13\sqrt{x} + 45}{+24x^3 + 22x^{5/2} + 22x^2 + 24x^{3/2} + 88x + 13\sqrt{x} + 45} \right)^2 - 144x \left(\frac{20x^6 + 4x^{11/2} + 9x^5 + 44x^{9/2} + 12x^4 + 32x^{7/2} + 14x^3 + 32x^{5/2} + 12x^2 + 44x^{3/2} + 9x + 4\sqrt{x} + 20}{+12x^2 + 44x^{3/2} + 9x + 4\sqrt{x} + 20} \right)^2.$$

After simplifications, we have

$$h_{42}(x) = 2(\sqrt{x} - 1)^4 \times v(x),$$

where

$$v(x) = \left(\begin{aligned} &2025x^{12} + 9270x^{23/2} + 14344x^{11} + \\ &+ 8634x^{21/2} + 27498x^{10} + \\ &+ 15106x^{19/2} + 9952x^9 - 2034x^{17/2} - \\ &- 9001x^8 - 9380x^{15/2} - 12776x^7 + \\ &+ 1444x^{13/2} - 36436x^6 + 1444x^{11/2} - \\ &- 12776x^5 - 9380x^{9/2} - 9001x^4 - \\ &- 2034x^{7/2} + 9952x^3 + 15106x^{5/2} + \\ &+ 27498x^2 + 8634x^{3/2} + \\ &+ 14344x + 9270\sqrt{x} + 2025 \end{aligned} \right).$$

Now we shall show that $v(x) > 0$. Let us consider

$$m(t) = v(t^2) = \left(\begin{aligned} &2025t^{24} + 9270t^{23} + 14344t^{22} + \\ &+ 8634t^{21} + 27498t^{20} + 15106t^{19} + \\ &+ 9952t^{18} - 2034t^{17} - 9001t^{16} - \\ &- 9380t^{15} - 12776t^{14} + 1444t^{13} - \\ &- 36436t^{12} + 1444t^{11} - 12776t^{10} - \\ &- 9380t^9 - 9001t^8 - 2034t^7 + \\ &+ 9952t^6 + 15106t^5 + 27498t^4 + \\ &+ 8634t^3 + 14344t^2 + 9270t + 2025 \end{aligned} \right).$$

The polynomial equation $m(t) = 0$ of 24^{th} degree admits 24 solutions. Out of them 22 are complex (not written here) and two of them are real given by

$$-1.125443752 \text{ and } -0.8885384079.$$

Both these solutions are negative. Since we are working with $t > 0$, this means that there are no real positive solutions of the equation $m(t) = 0$. Thus we conclude that either $m(t) > 0$ or $m(t) < 0$, for all $t > 0$. In order to check it is sufficient to see for any particular value of $m(t)$, for example when $t = 1$. This gives $m(1) = 73728$, hereby proving that $m(t) > 0$ for all $t > 0$, consequently, $v(x) > 0$, for all $x > 0$, proving that $h_{42}(x) > 0$, $\forall x > 0$, $x \neq 1$. Since $h_{42}(x) > 0$, proving that $k_{42}(x) > 0$. Also we have

$$\beta = \sup_{x \in (0, \infty)} g_{JM_3-F\Psi}(x) = \lim_{x \rightarrow 1} g_{JM_3-F\Psi}(x) = \frac{1}{24}.$$

Parts 1-55 refers to the proof of the inequalities given in (7) and the parts 56-59 give the proof of (8). Combining the parts 1-59 we get the proof of the Theorem 2.1. \square

2.1 Remark

1. Theorem 2.1 connects 54 members out of 55 appearing in the pyramid. Since some them are equals by multiplicative constants, the Theorem 2.1 contains 47 different measures. In this way we can make a sequential inequality connecting 34 divergence measures.
2. From the inequalities given in (7) and (8), it is interesting to observe that all the measures remain between $D_{I\Delta}^1$ and $D_{F\Psi}^{46}$, i.e., in between the first members of first and last line of the pyramid.
3. The last members of each line (corners members) of the pyramid are connected in an increasing order, i.e.,

$$\begin{aligned} D_{I\Delta}^1 &\leq \frac{8}{9} D_{M_1\Delta}^3 \leq \frac{8}{11} D_{M_2\Delta}^6 \leq \frac{2}{3} D_{h\Delta}^{10} \leq \\ &\leq \frac{8}{15} D_{M_3\Delta}^{15} \leq \frac{1}{2} D_{J\Delta}^{21} \leq \frac{1}{3} D_{T\Delta}^{28} \leq \\ &\leq \frac{1}{3} D_{K_0\Delta}^{36} \leq \frac{1}{6} D_{\Psi\Delta}^{45} \leq \frac{1}{9} D_{F\Delta}^{55}. \end{aligned} \quad (13)$$

3 Equivalent Inequalities

As a consequence of Theorem 2.1, the sequences of inequalities appearing in (7) and (8) can be written in an individual form. This means that the 59 results proving the Theorem 2.1 can be written in an equivalent form. This we have done below in two groups. The first group is with four measures in each case and the second group is with three measures.

Group 1

1. $80M_1 + 16M_3 \leq \Delta + 20h$;
2. $\Delta + 32h \leq 4T + 128M_1$;
3. $6\Delta + 256M_2 \leq 192I + 3K_0$;
4. $288M_1 + 224M_2 \leq 168I + 9J$;

5. $12M_1 + 20M_2 \leq 5I + 3T$;
6. $9J + 256M_2 \leq 192I + 72T$;
7. $10T + 32M_2 \leq 3J + 10h$;
8. $72I + 128T \leq 9K_0 + 512M_3$;
9. $8I + 4J \leq K_0 + 32h$;
10. $4\Delta + 8K_0 \leq \Psi + 64h$;
11. $16I + 10K_0 \leq \Psi + 10J$;
12. $26K_0 + 192M_1 \leq 3\Psi + 832M_3$;
13. $32M_1 + 32M_3 \leq J + 8T$;
14. $4\Delta + 3\Psi \leq F + 6K_0$;
15. $48I + 8\Psi \leq 3F + 128T$;
16. $48J + \Psi \leq 2F + 1536M_3$.

Group 2

1. $I \leq \frac{\Delta+128M_1}{36}$;
2. $I \leq \frac{4\Delta+K_0}{24}$;
3. $I \leq \frac{20\Delta+\Psi}{96}$;
4. $I \leq \frac{32\Delta+F}{144}$;
5. $M_1 \leq \frac{\Delta+24M_2}{88}$;
6. $M_1 \leq \frac{120I+K_0}{512}$;
7. $M_1 \leq \frac{624I+\Psi}{2560}$;
8. $M_1 \leq \frac{1008I+F}{4096}$;
9. $M_1 \geq \frac{3I+2M_2}{18}$;
10. $M_2 \leq \frac{\Delta+44h}{64}$;
11. $M_2 \leq \frac{T+26M_1}{10}$;
12. $M_2 \leq \frac{K_0+208M_1}{80}$;
13. $M_2 \leq \frac{\Psi+1184M_1}{416}$;
14. $M_2 \leq \frac{F+1952M_1}{672}$;
15. $M_2 \geq \frac{3I+9h}{16}$;
16. $h \leq \frac{\Delta+64M_3}{20}$;
17. $h \leq \frac{3J+128M_2}{120}$;
18. $h \leq \frac{T+16M_2}{13}$;
19. $h \leq \frac{K_0+128M_2}{104}$;
20. $h \leq \frac{\Psi+768M_2}{592}$;
21. $h \leq \frac{F+1280M_2}{976}$;
22. $h \geq \frac{3I+16M_3}{7}$;
23. $M_3 \leq \frac{2\Delta+15J}{512}$;
24. $M_3 \leq \frac{3J+8h}{128}$;
25. $M_3 \leq \frac{T+3h}{16}$;
26. $M_3 \leq \frac{K_0+24h}{128}$;
27. $M_3 \leq \frac{F+304h}{1280}$;
28. $M_3 \leq \frac{\Psi+176h}{768}$;
29. $J \leq \frac{K_0+16h}{3}$;
30. $J \leq \frac{2\Delta+16T}{3}$;
31. $J \leq \frac{\Psi+128h}{18}$;
32. $J \leq \frac{F+224h}{30}$;
33. $J \geq \frac{120T+256M_2}{39}$;
34. $J \geq \frac{8T+256M_3}{9}$;
35. $K_0 \leq \frac{6J+\Psi}{8}$;
36. $K_0 \leq \frac{12J+F}{14}$;
37. $K_0 \leq \frac{3\Psi+512M_3}{22}$;
38. $K_0 \leq \frac{3F+1024M_3}{38}$;
39. $\Psi \leq \frac{F+16T}{2}$.

Direct relations of the inequalities given in Groups 1 and 2 to the inequalities given in (5) shall be dealt elsewhere.

4 Reverse Inequalities

In view of Theorem 2.1, we shall derive some inequalities in reverse order for the last three lines of the **pyramid**.

1. Combining the inequalities given in the 10th line of the pyramid and the one given in (7) having the measure $F(P||Q)$, we have the following extended inequality

$$\begin{aligned} D_{F\Psi}^{46} &\leq D_{FK_0}^{47} \leq D_{FT}^{48} \leq D_{FJ}^{49} \leq D_{FM_3}^{50} \leq D_{Fh}^{51} \leq \\ &\leq D_{FM_2}^{52} \leq D_{FM_1}^{53} \leq D_{FI}^{54} \leq D_{F\Delta}^{55} \leq \frac{9}{8} D_{FI}^{54} \leq \\ &\leq \frac{8}{7} D_{FM_1}^{53} \leq \frac{72}{61} D_{FM_2}^{52} \leq \frac{6}{5} D_{Fh}^{51} \leq \left\{ \begin{array}{l} \frac{9}{7} D_{FJ}^{49} \\ \frac{24}{19} D_{FM_3}^{50} \end{array} \right\} \leq \\ &\leq \frac{3}{2} D_{FK_0}^{47} \leq \frac{3}{2} D_{FT}^{48} \leq 3 D_{F\Psi}^{46}. \end{aligned} \quad (14)$$

According to inequalities given in pyramid we have $D_{FJ}^{49} \leq D_{FM_3}^{50}$ but according to our approach we don't have *reverse relation* among the measures D_{FJ}^{49} and $D_{FM_3}^{50}$. Also D_{Fh}^{51} is related to D_{FJ}^{49} and $D_{FM_3}^{50}$ with different multiplicative constants. We call the expression (3.20) as *reverse inequalities*

2. Combining the inequalities given in the 9th line of the pyramid and the one given in (7) having the measure $\Psi(P||Q)$, we have the following extended inequality

$$\begin{aligned} D_{\Psi K_0}^{37} &\leq D_{\Psi T}^{38} \leq D_{\Psi J}^{39} \leq D_{\Psi M_3}^{40} \leq D_{\Psi h}^{41} \leq D_{\Psi M_2}^{42} \leq \\ &\leq D_{\Psi M_1}^{43} \leq D_{\Psi I}^{44} \leq D_{\Psi \Delta}^{45} \leq \frac{6}{5} D_{\Psi I}^{44} \leq \frac{16}{15} D_{\Psi M_1}^{43} \leq \\ &\leq \frac{48}{37} D_{\Psi M_2}^{42} \leq \frac{4}{3} D_{\Psi h}^{41} \leq \left\{ \begin{array}{l} \frac{3}{2} D_{\Psi J}^{39} \\ \frac{16}{11} D_{\Psi M_3}^{40} \end{array} \right\} \leq \\ &\leq 2 D_{\Psi K_0}^{37} \leq 2 D_{\Psi T}^{38}. \end{aligned} \quad (15)$$

According to inequalities given in pyramid we have $D_{\Psi J}^{39} \leq D_{\Psi M_3}^{40}$ but according to our approach we don't have *reverse relation* among the measures $D_{\Psi J}^{39}$ and $D_{\Psi M_3}^{40}$. Also $D_{\Psi h}^{41}$ is related to $D_{\Psi J}^{39}$ and $D_{\Psi M_3}^{40}$ with different multiplicative constants. Again we call the expression (15) as *reverse inequalities*

3. Combining the inequalities given in the 8th line of the pyramid and the one given in (7) having the measure $\Psi(P||Q)$, we have the following extended inequality

$$\begin{aligned} D_{K_0 T}^{29} &\leq D_{K_0 J}^{30} \leq D_{K_0 M_3}^{31} \leq D_{K_0 h}^{32} \leq \\ &\leq D_{K_0 M_2}^{33} \leq D_{K_0 M_1}^{34} \leq D_{K_0 I}^{35} \leq D_{K_0 \Delta}^{36} \leq \\ &\leq \frac{3}{2} D_{K_0 I}^{35} \leq \frac{8}{5} D_{K_0 M_1}^{34} \leq \frac{24}{13} D_{K_0 M_2}^{33} \leq \\ &\leq 2 D_{K_0 h}^{32} \leq \left\{ \begin{array}{l} 3 D_{K_0 J}^{30} \\ \frac{8}{3} D_{K_0 M_3}^{31} \end{array} \right\}. \end{aligned} \quad (16)$$

We observe that the measure $D_{K_0 T}^{29}$ don't appears in the reverse side. Moreover, it don't appears in Theorem 2.1 too.

Similarly we can write *reverse inequalities* for the other lines of the pyramid.

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